

Fedosov \ast -products and quantum momentum maps

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Abstract

The purpose of the paper is to study various aspects of star products on a symplectic manifold related to the Fedosov method.

By introducing the notion of “quantum exponential maps”, we give a criterion characterizing Fedosov connections. As a consequence, a geometric realization is obtained for the equivalence between an arbitrary \ast -product and a Fedosov one.

Every Fedosov \ast -product is shown to be a Vey \ast -product. Consequently, one obtains that every \ast -product is equivalent to a Vey \ast -product, a classical result of Lichnerowicz.

Quantization of a hamiltonian G -space, and in particular, quantum momentum maps are studied. Lagrangian submanifolds are also studied under a deformation quantization.

1 Introduction

In classical mechanics, observables are smooth functions on a phase space, which consist of a Poisson algebra, while in quantum mechanics observables become a noncommutative associative algebra. Deformation quantization, as laid out by Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer in 1970's [5], is one of the important attempts aiming to establish a correspondence principle between these two mechanics. A classical phase space is usually a symplectic manifold M . A deformation quantization, or more precisely a star-product is a family of associative multiplication \ast_{\hbar} (depending on the Planck constant \hbar) on $C^\infty(M)[[\hbar]]$, the space of formal power series with coefficients in $C^\infty(M)$:

$$f \ast_{\hbar} g = fg - \frac{i\hbar}{2}\{f, g\} + \cdots + \hbar^k C_k(f, g) + \cdots, \quad \forall f, g \in C^\infty(M)[[\hbar]] \quad (1)$$

such that

$$(i). \quad C_k(f, g) = (-1)^k C_k(g, f);$$

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- (ii). $C_k(1, f) = C_k(f, 1) = 0$, for $k \geq 1$;
- (iii). each $C_k(\cdot, \cdot)$ is a bidifferential operator.

Here $\{f, g\}$ is the Poisson bracket on the symplectic manifold M .

Such a $*$ -product always exists on a symplectic vector space V , which is known as Moyal-Weyl formula:

$$f *_h g = \sum_{k=0}^{\infty} \left(-\frac{i\hbar}{2}\right)^k \frac{1}{k!} \pi^{i_1 j_1} \dots \pi^{i_k j_k} \frac{\partial^k f}{\partial y^{i_1} \dots \partial y^{i_k}} \frac{\partial^k g}{\partial y^{j_1} \dots \partial y^{j_k}}, \quad \forall f, g \in C^\infty(V)[[\hbar]], \quad (2)$$

where y^1, \dots, y^{2n} are linear coordinates on V , and $\pi^{ij} = \{y^i, y^j\}$. It is simple to see that this definition is independent of the choice of linear coordinates.

Moyal-Weyl formula has a straightforward generalization to the case of a symplectic manifold admitting a flat torsion free symplectic connection ∇ , as shown in [5]. In this case, for any $f, g \in C^\infty(M)[[\hbar]]$, set

$$(f *_h g)(x) = [(exp_x^* f)(y) *_h (exp_x^* g)(y)]|_{y=0}, \quad (3)$$

where $exp_x : T_x M \rightarrow M$ is the exponential map, defined in a neighborhood of the origin, corresponding to the connection ∇ , and the $*_h$ -product on the RHS refers to the standard Moyal-Weyl $*$ -product on the symplectic vector space $T_x M$. More explicitly, one has

$$(f *_h g)(x) = \sum_{k=0}^{\infty} \left(-\frac{i\hbar}{2}\right)^k \frac{1}{k!} \pi^{i_1 j_1} \dots \pi^{i_k j_k} (\partial_{i_1} \dots \partial_{i_k} f)(\partial_{j_1} \dots \partial_{j_k} g). \quad (4)$$

However, the multiplication defined by Equation (3) fails to be associative when ∇ has a curvature. The existence proof of $*$ -products on a general symplectic manifold was first obtained by de Wilde and Lecomte [8] using a homological argument. Later, an alternate proof using Weyl manifolds was found by Omori et al.[21]. Guillemin showed that their results in [15] also implies the existence of $*$ -products.

Recently, Fedosov has given a nice geometrical existence proof [11] [12], which provides a useful tool for understanding $*$ -products geometrically. This paper grew out from an attempt to understand the Fedosov method, as well as an attempt to solve some other related problems using such a method.

Roughly speaking, Fedosov's method is to make a "quantum correction" to Equation (3) when the curvature exists. As discussed early in this introduction, each tangent space $T_x M$ is a symplectic vector space, hence can be quantized using the standard Moyal-Weyl product. This enables us to obtain a bundle of algebras $W \rightarrow M$, called Weyl bundle by Fedosov, which can be thought of as a kind of "quantum tangent bundle". Fedosov found a nice iteration method of constructing a flat connection on the Weyl bundle, whose parallel sections can be naturally identified with $C^\infty(M)[[\hbar]]$. Thus the product on the bundle induces a $*$ -product on $C^\infty(M)[[\hbar]]$. Intuitively, Fedosov connection can be thought of as a "quantum connection" on the "quantum tangent bundle" W , which is indeed obtained by adding some "quantum correction" to the usual affine connection on the tangent bundle (see Section 2). Then, the correspondence between $C^\infty(M)[[\hbar]]$ and its parallel sections can be considered as taking the exponential map for such a quantum connection.

This viewpoint has already been pointed out by Weinstein (see [25] [9]), and is essential to our introduction of the notion of “quantum exponential maps”.

By a “quantum exponential map”, we mean a map from $C^\infty(M)[[\hbar]]$ to the space of sections ΓW of the Weyl bundle, which satisfies certain axioms (see Section 3 for details). In fact, we will show that any quantum exponential map is equivalent to a Fedosov connection, which is what is expected: connections and exponential maps are equivalent. This result will enable us to characterize those subalgebras of ΓW arising from Fedosov connections. As an application, we will realize geometrically the equivalence between any given $*$ -product and a Fedosov one, a result proved by several authors including Nest-Tsygan [20], Deligne [10] and recently Bertelson-Cahen-Gutt [7].

Since a “quantum exponential map” is a “quantum correction” to the usual exponential map, the $*$ -products obtained by Fedosov method, called Fedosov $*$ -products in the paper, should be closely related to Equation (3). A Vey $*$ -product, by definition, is a $*$ -product where the principal terms coincide with those as in Equation (3) (see Section 4 for the precise definition). Vey $*$ -products have played an important role since the beginning of deformation quantization theory (see [5] [18] [22]). In this paper, we will show that every Fedosov $*$ -product is a Vey $*$ -product. As a consequence, we recover the following well known result of Lichnerowicz [18]: any $*$ -product is equivalent to a Vey $*$ -product.

Because of the basic simplicity of its construction, Fedosov method provides us a useful tool for studying some other problems in deformation quantization theory. In this paper, in particular, we will study the following question:

What do lagrangian submanifolds correspond under a deformation quantization?

Given a lagrangian submanifold L , the space $C_L^\infty(M)$ of smooth functions vanishing on L forms a Poisson subalgebra. For any $*$ -product, we will show that under some “quantum correction” $C_L^\infty(M)[[\hbar]]$ becomes a subalgebra.

In symplectic geometry, hamiltonian G -spaces, and in particular, momentum maps play a very special role. It is natural to ask: what is the quantum analogue of momentum maps? The second part of the paper, as another application of Fedosov method, is devoted to the study of quantum momentum maps. In particular, we will derive a sufficient condition for their existence. When a quantum momentum map exists, we obtain a pair of mutual commutants which can be considered as a quantum analogue of the well known Poisson dual pair $\mathfrak{g}^* \xleftarrow{J} M \xrightarrow{pr} M/G$ of Weinstein [24].

Derivations are always important for associative algebras. In Appendix A, we will collect some basic results regarding derivations on a $*$ -algebra $C^\infty(M)[[\hbar]]$, which are needed for the study of quantum momentum maps. Most of them are proved using Fedosov method. Even for some well known results, we will see that Fedosov method provides a simpler way in understanding them.

While the paper is in writing, it came to the author’s attention that quantum momentum maps are also being under study by some other authors including Astashkevich [4], Kostant and Tsygan. In particular, Theorem 6.4 has also been proved by Kostant.

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2 Fedosov quantization

In this section, we will recall some basic ingredients of Fedosov construction of $*$ -products on a symplectic manifold, as well as some useful notations, which will be needed in the future. For details, readers should consult [11] [12].

Let (M, ω) be a symplectic manifold of dimension $2n$. Then, each tangent space $T_x M$ is equipped with a linear symplectic structure, which can be quantized by the standard Moyal-Weyl product. The resulting space is denoted by W_x . More precisely,

Definition 2.1 *A formal Weyl algebra W_x associated to $T_x M$ is an associative algebra with a unit over \mathbb{C} , whose elements consist of formal power series in the formal parameter \hbar with coefficients being formal polynomials in $T_x M$. In other words, each element has the form*

$$a(y, \hbar) = \sum \hbar^k a_{k, \alpha} y^\alpha \quad (5)$$

where $y = (y^1, \dots, y^{2n})$ is a linear coordinate on $T_x M$, $\alpha = (\alpha_1, \dots, \alpha_{2n})$ is a multi-index and $y^\alpha = (y^1)^{\alpha_1} \dots (y^{2n})^{\alpha_{2n}}$. The product is defined by the Moyal-Weyl rule:

$$a * b = \sum_{k=0}^{\infty} \left(-\frac{i\hbar}{2}\right)^k \frac{1}{k!} \pi^{i_1 j_1} \dots \pi^{i_k j_k} \frac{\partial^k a}{\partial y^{i_1} \dots \partial y^{i_k}} \frac{\partial^k b}{\partial y^{j_1} \dots \partial y^{j_k}}. \quad (6)$$

Let $W = \cup_{x \in M} W_x$. Then W is a bundle of algebras over M , called the Weyl bundle over M . Its space of sections ΓW forms an associative algebra with unit under the fiberwise multiplication. One may think of W as a “quantum tangent bundle” of M , whose space of sections ΓW gives rise to a deformation quantization for the tangent bundle TM considered as a Poisson manifold, with fiberwise linear symplectic structure on $T_x M$, $x \in M$.

The center $Z(W)$ of ΓW consists of sections not containing y 's, thus can be naturally identified with $C^\infty(M)[[\hbar]]$.

By assigning degrees to y 's and \hbar with $\deg y^i = 1$ and $\deg \hbar = 2$, there is a natural filtration

$$\Gamma(W) \subset \Gamma(W_1) \subset \dots \Gamma(W_i) \subset \Gamma(W_{i+1}) \dots$$

with respect to the total degree $2k + l$ of the series terms in Equation (5).

A differential form with values in W is a section of the bundle $W \otimes \wedge^q T^* M$, which can be expressed locally as

$$a(x, y, \hbar, dx) = \sum \hbar^k a_{k, i_1 \dots i_p, j_1 \dots j_q} y^{i_1} \dots y^{i_p} dx^{j_1} \wedge \dots \wedge dx^{j_q}. \quad (7)$$

Here the coefficient $a_{k,i_1 \dots i_p, j_1 \dots j_q}$ is a covariant tensor symmetric with respect to $i_1 \dots i_p$ and antisymmetric in $j_1 \dots j_q$. For short, we denote the space of sections of the bundle $W \otimes \wedge^q T^*M$ by $\Gamma W \otimes \Lambda^q$.

The usual exterior derivative on differential forms extends, in a straightforward way, to an operator δ on W -valued differential forms:

$$\delta a = dx^i \wedge \frac{\partial a}{\partial y^i}, \quad \forall a \in \Gamma W \otimes \Lambda^*. \quad (8)$$

By δ^{-1} , we denote its “inverse” operator as defined by:

$$\delta^{-1} a = \frac{1}{p+q} y^i \left(\frac{\partial}{\partial x^i} \lrcorner a \right) \quad (9)$$

when $p+q > 0$, and $\delta^{-1} a = 0$ when $p+q = 0$. Here $a \in \Gamma W \otimes \Lambda^q$ is homogeneous of degree p in y .

There is a “Hodge”-decomposition:

$$a = \delta \delta^{-1} a + \delta^{-1} \delta a + a_{00}, \quad \forall a \in \Gamma W \otimes \Lambda^*, \quad (10)$$

where $a_{00}(x)$ is the constant term of a , i.e, the 0-form term of $a|_{y=0}$ or $a_{00}(x) = a(x, 0, 0, 0)$. The operator δ resembles most basic properties of the usual exterior derivatives. For example,

$$\delta^2 = 0 \quad \text{and} \quad (\delta^{-1})^2 = 0.$$

Let ∇ be a torsion-free symplectic connection on M , and $\partial : \Gamma W \longrightarrow \Gamma W \otimes \Lambda^1$ its induced covariant derivative.

Consider a connection on W of the form:

$$D = -\delta + \partial + \frac{i}{\hbar} [\gamma, \cdot], \quad (11)$$

with $\gamma \in \Gamma W \otimes \Lambda^1$.

Clearly, D is a derivation with respect to the Moyal-Weyl product, i.e.,

$$D(a * b) = a * Db + Da * b. \quad (12)$$

A simple calculation yields that

$$D^2 a = -\left[\frac{i}{\hbar} \Omega, a\right], \quad \forall a \in \Gamma W, \quad (13)$$

where

$$\Omega = \omega - R + \delta \gamma - \partial \gamma - \frac{i}{\hbar} \gamma^2. \quad (14)$$

Here $R = \frac{1}{4} R_{ijkl} y^i y^j dx^k \wedge dx^l$ and $R_{ijkl} = \omega_{im} R_{jkl}^m$ is the curvature tensor of the symplectic connection.

A connection of the form (11) is called *Abelian* if Ω is a scalar 2-form, i.e., $\Omega \in \Omega^2(M)[[\hbar]]$. It is called a *Fedosov connection* if it is Abelian and in addition $\gamma \in \Gamma W_3 \otimes \Lambda^1$. For an Abelian connection, the Bianchi identity implies that $d\Omega = D\Omega = 0$, i.e., $\Omega \in Z^2(M)[[\hbar]]$. In this case, Ω is called the Weyl curvature.

Theorem 2.2 (Fedosov) *Let ∂ be any torsion free symplectic connection, and $\Omega = \omega + \hbar\omega_1 + \dots \in Z^2(M)[[\hbar]]$ a perturbation of the symplectic form in the space $Z^2(M)[[\hbar]]$. There exists a unique $\gamma \in \Gamma W_3 \otimes \Lambda^1$ such that D , given by Equation (11), is a Fedosov connection, which has Weyl curvature Ω and satisfies*

$$\delta^{-1}\gamma = 0.$$

Proof. It suffices to solve the equation:

$$\omega - R + \delta\gamma - \partial\gamma - \frac{i}{\hbar}\gamma^2 = \Omega, \quad (15)$$

which is equivalent to

$$\delta\gamma = \tilde{\Omega} + \partial\gamma + \frac{i}{\hbar}\gamma^2, \quad (16)$$

where $\tilde{\Omega} = \Omega - \omega + R$. Applying the operator δ^{-1} to Equation (16) and using the Hodge decomposition, we obtain

$$\gamma = \delta^{-1}\tilde{\Omega} + \delta^{-1}(\partial\gamma + \frac{i}{\hbar}\gamma^2). \quad (17)$$

Here we note that $\gamma_{00} = 0$ since γ is a 1-form.

Since the operator ∂ preserves the filtration and δ^{-1} raises it by 1, the iteration formula (17) has a unique solution. Moreover since $\delta^{-1}\tilde{\Omega}$ is at least of degree 3, the solution γ is indeed in $\Gamma W_3 \otimes \Lambda^1$.

□

Remark The theorem above indicates that a Fedosov connection is uniquely determined by a torsion free symplectic connection ∇ and a Weyl curvature $\Omega = \sum_i \hbar^i \omega_i \in Z^2(M)[[\hbar]]$. For this reason, we will say that the connection D defined above is a Fedosov connection corresponding to the pair (∇, Ω) .

If D is a Fedosov connection, the space of all parallel sections W_D automatically becomes an associative algebra. Fedosov proved that W_D can be naturally identified with $C^\infty(M)[[\hbar]]$, and therefore induces a $*$ -product on $C^\infty(M)[[\hbar]]$, which we will call a *Fedosov $*$ -product*.

Let σ denote the projection from W_D to its center $C^\infty(M)[[\hbar]]$ defined as $\sigma(a) = a|_{y=0}$.

Theorem 2.3 (Fedosov) *σ establishes an isomorphism between W_D and $C^\infty(M)[[\hbar]]$ as vector spaces. Therefore, it induces an associative algebra structure on $C^\infty(M)[[\hbar]]$, which is a $*$ -product.*

Proof. To prove that the two vector spaces are isomorphic, it suffices to show that for any $a_0(x, \hbar) \in C^\infty(M)[[\hbar]]$ there is a unique section $a \in W_D$ such that $\sigma(a) = a_0$.

The equation $Da = 0$ can be written as

$$\delta a = \partial a + [\frac{i}{\hbar}\gamma, a].$$

Applying the operator δ^{-1} , it follows from the Hodge decomposition that

$$a = a_0 + \delta^{-1}(\partial a + [\frac{i}{\hbar}\gamma, a]). \quad (18)$$

By the iteration method, we see that the equation above has a unique solution since δ^{-1} increases the filtration. The rest of the claim can be easily verified.

□

3 Quantum exponential maps

If ∇ is flat and $\Omega = \omega$, the Fedosov connection is simply given by $D = -\delta + \partial$. In this case, the solution to Equation (18) can be expressed explicitly as

$$a = \sum_{k=0}^{\infty} \frac{1}{k!} (\partial_{i_1} \cdots \partial_{i_k} a_0) y^{i_1} \cdots y^{i_k},$$

which is just the Taylor expansion of $\exp_x^* a_0$ at the origin. So the correspondence from $C^\infty(M)[[\hbar]]$ to W_D is indeed the pullback of the (C^∞ -jet at the origin of the) usual exponential map. Thus for a general Fedosov connection, one may consider the correspondence $C^\infty(M)[[\hbar]] \longrightarrow W_D$ as a “quantum correction” to the exponential map. In this section, we will make this idea more precise by introducing the notion of quantum exponential maps, which gives a simple characterization for Fedosov connections. As an application, we will realize geometrically the equivalence between an arbitrary $*$ -product and one from Fedosov method, namely a Fedosov $*$ -product.

Definition 3.1 *A quantum exponential map is an \hbar -linear map $\rho : C^\infty(M)[[\hbar]] \longrightarrow \Gamma W$ such that*

- (i). $\rho(C^\infty(M)[[\hbar]])$ is a subalgebra of ΓW ;
- (ii). $\rho(a)|_{y=0} = a, \quad \forall a \in C^\infty(M)[[\hbar]]$;
- (iii). $\rho(a) = a + \delta^{-1} da, \quad \forall a \in C^\infty(M), \quad \text{mod } W_2$;
- (iv). $\rho(a)$ can be expressed as a formal power series in y and \hbar , with coefficients being derivatives of a .

Given a quantum exponential map ρ , the Condition (ii) implies that ρ establishes an isomorphism between $C^\infty(M)[[\hbar]]$ and its image as vector spaces. Therefore, $C^\infty(M)[[\hbar]]$ becomes an associative algebra because of the first condition. It is simple to see that the third condition implies that this is indeed a deformation of the symplectic structure, and the last one implies that it is a $*$ -product.

Clearly, for any Fedosov connection D , the map from $C^\infty(M)[[\hbar]]$ to W_D , as constructed by Fedosov (see Theorem 2.3), satisfies all the properties of a quantum exponential map. So quantum exponential map always exists, and one may consider Fedosov construction as a way of constructing a quantum exponential map. In what follows, we will show that the converse is also true. That is,

Theorem 3.2 *Quantum exponential maps are equivalent to Fedosov connections.*

Before proving this theorem, we start with investigating the following closely related question: what kind of subalgebras of ΓW arises from a Fedosov connection?

Proposition 3.3 *Suppose that $\mathcal{A} \subseteq \Gamma W$ is a subalgebra satisfying the following conditions:*

- (i). “completeness”- for any $x_0 \in M$, and any $a(y, \hbar) \in W_{x_0}$, there is an element $\tilde{a}(x, y, \hbar) \in \mathcal{A}$ such that $\tilde{a}(x_0, y, \hbar) = a(y, \hbar)$.
- (ii). “uniqueness”- if \tilde{a} and $\tilde{b} \in \mathcal{A}$ such that $\tilde{a}|_{x_0} = \tilde{b}|_{x_0}$, then $\tilde{a}_*|_{x_0} = \tilde{b}_*|_{x_0}$, where \tilde{a} and \tilde{b} are considered as maps: $M \longrightarrow W$, and \tilde{a}_* and \tilde{b}_* refer to their derivatives.

Then, there exists a unique Abelian connection D such that $\mathcal{A} \subseteq W_D$.

Proof. Take a torsion free symplectic connection ∇ , and let $\partial : \Gamma W \longrightarrow \Gamma W \otimes \Lambda^1$ be the corresponding covariant derivative. For any $x_0 \in M$, introduce an operator $\rho_{x_0} : W_{x_0} \longrightarrow W_{x_0} \otimes \Lambda^1$ by

$$\rho_{x_0}(a(y, \hbar)) = (\delta - \partial)\tilde{a}(x, y, \hbar)|_{x=x_0},$$

where $\tilde{a}(x, y, \hbar) \in \mathcal{A}$ such that $\tilde{a}(x_0, y, \hbar) = a(y, \hbar)$. By assumption, the map ρ_{x_0} is well-defined and is in fact a derivation of the algebra W_{x_0} . Therefore, there is a unique element $\gamma_{x_0} \in W_{x_0} \otimes \Lambda^1$ with $\gamma_{x_0}|_{y=0} = 0$ such that $\rho_{x_0} = ad_{\frac{i}{\hbar}\gamma_{x_0}} = [\frac{i}{\hbar}\gamma_{x_0}, \cdot]$. Applying this process pointwisely, we obtain a global section $\gamma \in W_1 \otimes \Lambda^1$ with $\gamma_0 \stackrel{def}{=} \gamma|_{y=0} = 0$. Let $D = -\delta + \partial + [\frac{i}{\hbar}\gamma, \cdot]$. Then, D is a connection on the Weyl bundle W and satisfies the condition: $D\tilde{a}(x, y, \hbar) = 0$ for all $\tilde{a}(x, y, \hbar) \in \mathcal{A}$. As in Section 2, let $\Omega = \omega - R + \delta\gamma - \partial\gamma - \frac{i}{\hbar}\gamma^2$ denote the Weyl curvature of the connection D . It thus follows that

$$D^2\tilde{a} = [\frac{i}{\hbar}\Omega, \tilde{a}] = 0, \quad \forall \tilde{a} \in \mathcal{A}.$$

Since \mathcal{A} is complete, Ω belongs to the center. Therefore, $\Omega \in Z^2(M)[[\hbar]]$. In other words, D is an Abelian connection.

□

The following lemma gives a simple sufficient condition for a subalgebra $\mathcal{A} \subseteq \Gamma W$ being complete.

Lemma 3.4 *Let $\mathcal{A} \subseteq \Gamma W$ be a subalgebra with unit. Suppose that for any $a_0(x) \in C^\infty(M)$, there is $\tilde{a} \in \mathcal{A}$ such that*

$$\tilde{a} = a_0 + \delta^{-1}da_0, \quad \text{mod } W_2.$$

Then, \mathcal{A} is complete.

Proof. We will prove, by induction, the following statement: $\forall x_0 \in M$ and $a(y, \hbar) \in W_k|_{x_0}$, there is an element $\tilde{a} \in \mathcal{A}$ such that $\tilde{a}(x_0, y, \hbar) - a(y, \hbar) \in W_{k+1}|_{x_0}$. The conclusion will then follow immediately from an iteration argument.

By assumption, the statement holds for both $k = 0$ and $k = 1$.

Assume that $a(y, \hbar) = \hbar^j y^{i_1} \cdots y^{i_p}$ with $2j + p = k$. Now

$$\begin{aligned} & a(y, \hbar) - \hbar^j y^{i_1} * \cdots * y^{i_p} \\ &= \hbar^j (y^{i_1} \cdots y^{i_p} - y^{i_1} * \cdots * y^{i_p}) \\ &= \sum_{2i+s=p} \hbar^{i+j} C_{i+j, j_1 \dots j_s} y^{j_1} \cdots y^{j_s}, \end{aligned}$$

where $i \geq 1$, and therefore $s = p - 2i = k - 2j - 2i < k$.

Applying the induction assumption for each term $y^{j_1} \cdots y^{j_s}$ in the above summation, we conclude that there is an element $\tilde{b} \in \mathcal{A}$ such that

$$a(y, \hbar) - \hbar^j y^{i_1} * \cdots * y^{i_p} = \tilde{b}(x_0, y, \hbar), \quad \text{mod } W_{k+1}.$$

On the other hand, for each y^{i_l} , there is $\tilde{a}^{i_l} \in \mathcal{A}$ such that $\tilde{a}^{i_l}|_{x_0} = y^{i_l}$, mod W_2 , for $1 \leq l \leq p$. It is then clear that

$$\hbar^j y^{i_1} * \cdots * y^{i_p} = \hbar^j \tilde{a}^{i_1} * \cdots * \tilde{a}^{i_p}|_{x_0}, \quad \text{mod } W_{k+1}.$$

Thus, we have

$$a(y, \hbar) = \hbar^j \tilde{a}^{i_1} * \cdots * \tilde{a}^{i_p}|_{x_0} + \tilde{b}(x_0, y, \hbar), \quad \text{mod } W_{k+1}.$$

This concludes the proof. □

Now we are ready to formulate the following result, which gives a criterion characterizing a Fedosov algebra.

Theorem 3.5 *Let \mathcal{A} be a subalgebra of ΓW satisfying:*

(i). $\forall a_0 \in C^\infty(M)$, there is an element $\tilde{a} \in \mathcal{A}$ such that

$$\tilde{a} = a_0 + \delta^{-1} da_0, \quad \text{mod } W_2;$$

(ii). if a and $b \in \mathcal{A}$ such that $a|_{x_0} = b|_{x_0}$, then $a_*|_{x_0} = b_*|_{x_0}$, where a and b are considered as maps: $M \longrightarrow W$, and a_* and b_* refer to their derivatives.

Then, \mathcal{A} is a Fedosov algebra, i.e. \mathcal{A} arises from a Fedosov connection.

Proof. According to Lemma 3.4, \mathcal{A} is complete. Thus by Proposition 3.3, there is an Abelian connection D such that $\mathcal{A} \subseteq W_D$.

Lemma 3.6 *This Abelian connection D is in fact a Fedosov connection.*

Proof. By assumption, for any $a_0 \in C^\infty(M)$, there is $\tilde{a} \in \mathcal{A}$ such that

$$\tilde{a} = a_0 + \delta^{-1} da_0, \quad \text{mod } W_2.$$

It follows from $D\tilde{a} = 0$ that

$$(\delta - \partial)\tilde{a} = [\frac{i}{\hbar}\gamma, \tilde{a}].$$

The degree zero term of the LHS is easily seen to be zero, while the degree zero term of the RHS is $\{\gamma_1, \delta^{-1} da_0\}$, where γ_1 is the degree one term of γ . It thus follows that $\{\gamma_1, \delta^{-1} da_0\} = 0$. Since a_0 is arbitrary, γ_1 must be constant with respect to y . However it is linear in y , it has to be identically zero. This implies that $\gamma \in W_2 \otimes \Lambda^1$.

Denote by γ_2 the degree 2 term of γ , and assume that $\gamma_2 = \sum r_{ij,k}(x) y^i y^j dx^k$. Note that this is the most general form since $\gamma_0 = 0$. Also, we may always assume that $r_{ij,k} = r_{ji,k}$. Since D is Abelian, its curvature $\Omega \in Z^2(M)[[\hbar]]$. Assume that

$$\Omega = \sum_{i=0}^{\infty} \hbar^i \omega_i = \omega_0 + \hbar \omega_1 + \hbar^2 \omega_2 + \cdots. \quad (19)$$

On the other hand, according to Equation (14), we have

$$\Omega = \omega - R + \delta\gamma - \partial\gamma - \frac{i}{\hbar}\gamma^2. \quad (20)$$

Comparing the degree zero terms of Equations (19) and (20), it follows immediately that $\omega_0 = \omega$. Now the terms R , $\partial\gamma$ and $\frac{i}{\hbar}\gamma^2$ are all of degree not less than 2, so the only degree 1 term in Equation (20) would be $\delta\gamma_2$. Hence $\delta\gamma_2 = 0$. On the other hand, a simple calculation yields that

$$\begin{aligned} \delta\gamma_2 &= \sum (r_{ij,k}(x) y^j dx^i \wedge dx^k + r_{ij,k}(x) y^i dx^j \wedge dx^k) \\ &= \sum r_{ij,k}(x) y^j dx^i \wedge dx^k + \sum r_{ji,k}(x) y^j dx^i \wedge dx^k \\ &= 2 \sum r_{ij,k}(x) y^j dx^i \wedge dx^k. \end{aligned}$$

It thus follows that for any $i \neq k$,

$$\sum_j r_{ij,k}(x) y^j = \sum_j r_{kj,i}(x) y^j.$$

That is, $r_{ij,k}(x) = r_{kj,i}(x)$. Hence, $r_{ij,k}$ is completely symmetric with respect to i, j, k .

Let $\Gamma'_{ijk} = \Gamma_{ijk} + 2r_{ij,k}$. Then, Γ'_{ijk} defines a torsion free symplectic connection with induced differential $\partial' = d + [\frac{i}{\hbar}\Gamma', \cdot]$, where $\Gamma' = \frac{1}{2}\Gamma'_{ijk} y^i y^j dx^k$. It is easy to see that $\partial' = \partial + [\frac{i}{\hbar}\gamma_2, \cdot]$. Let $\gamma' = \gamma - \gamma_2$. Then $\gamma' \in W_3 \otimes \Lambda^1$, and $D = -\delta + \partial' + [\frac{i}{\hbar}\gamma', \cdot]$. This shows that D is indeed a Fedosov connection.

Lemma 3.7 *For any $\tilde{a} \in W_D$, there is $\tilde{a}_0 \in \mathcal{A}$ and $\tilde{b} \in W_D$ such that*

$$\tilde{a} = \tilde{a}_0 + \hbar \tilde{b}.$$

Proof. Let $a = \tilde{a}|_{y=0} = a_0(x) + \hbar a_1(x) + \cdots \in C^\infty(M)[[\hbar]]$. Take $\tilde{a}_0 \in \mathcal{A}$ such that $\tilde{a}_0 = a_0 + \delta^{-1}da_0 \mod W_2$, which is always possible by assumption. Thus, $\tilde{a}_0|_{y=0} = a_0(x) + O(\hbar)$ and then $(\tilde{a} - \tilde{a}_0)|_{y=0} = O(\hbar)$. However, we know that $\tilde{a} - \tilde{a}_0 \in W_D$ since $\tilde{a}_0 \in \mathcal{A} \subseteq W_D$. This implies that $\tilde{a} - \tilde{a}_0 = \hbar \tilde{b}$ for some $\tilde{b} \in W_D$. This concludes the proof of the lemma.

□

By using the lemma above repeatedly, one immediately obtains the other inclusion, i.e., $W_D \subseteq \mathcal{A}$. This concludes our proof of theorem 3.5.

□

Clearly, Theorem 3.2 is an immediate consequence of Theorem 3.5. As another application, we will give a geometric constructive proof for the following (see [7] [10] [20]):

Theorem 3.8 *Any *-product is equivalent to a Fedosov *-product.*

Proof. As in Section 2, let $P = TM$ be the regular Poisson manifold equipped with the fiberwise linear symplectic structures. Let $\mathcal{P} = \Gamma(\cup_{m \in M} \Gamma_0^\infty(T_m M, \mathbb{R}))$, where $\Gamma_0^\infty(T_m M, \mathbb{R})$ denotes the set of ∞ -jets at 0 of real valued functions on $T_m M$. Thus, \mathcal{P} is a Poisson algebra with a naturally induced Poisson structure.

Lemma 3.9 *Any *-product on $C^\infty(M)[[\hbar]]$ induces a *-product on the Poisson manifold $P = TM$ so that there is an algebra embedding $\rho : C^\infty(M)[[\hbar]] \longrightarrow \mathcal{P}[[\hbar]]$.*

Proof. Take any torsion-free symplectic connection ∇ , and for any fixed $m \in M$, let Exp_m be the formal symplectic exponential map introduced by Emmerich and Weinstein [9]. Then, $(Exp_m)^* : C^\infty(M) \longrightarrow \Gamma_0^\infty(T_m M, \mathbb{R})$ is a Poisson algebra morphism, which in fact maps $jet_m^\infty C^\infty(M)$ to $\Gamma_0^\infty(T_m M, \mathbb{R})$ isomorphically. Therefore, any *-product on $C^\infty(M)[[\hbar]]$ induces a *-product on $\Gamma_0^\infty(T_m M, \mathbb{R})[[\hbar]]$, hence on $C^\infty(T_m M)[[\hbar]]$. Thus, we obtain a regular *-product (see Appendix B for the definition), denoted as $\tilde{*}_h$, on the Poisson manifold P , and Exp^* is clearly an embedding of the algebra.

□

According to Proposition 9.1, a regular *-product on P is essentially unique. Hence, there exists an equivalence operator: $T_h = 1 + \hbar T_1 + \hbar^2 T_2 + \cdots$, with T_i being leafwise differential operators, between $(\mathcal{P}[[\hbar]], \tilde{*}_h)$ and the standard Weyl quantization ΓW . Let $\mathcal{A} = T_h Exp^*(C^\infty(M)[[\hbar]]) \subset \Gamma W$. Then \mathcal{A} is a subalgebra of ΓW . It is simple to see that \mathcal{A} satisfies all the conditions in Theorem 3.5, so it is a Fedosov algebra. This concludes the proof of the theorem.

□

Since every $*$ -product is equivalent to a Fedosov $*$ -product, its *characteristic class*, as defined by Nest-Tsygan [20], is the class in $H^2(M)[[\hbar]]$ of the Weyl curvature of its equivalent Fedosov $*$ -product. This is well-defined since the Weyl curvatures of equivalent Fedosov $*$ -products are cohomologous (see [12]). In fact, Fedosov showed that two Fedosov $*$ -products are equivalent iff their Weyl curvatures are cohomologous [12]. Thus (see [20]),

Theorem 3.10 *Two $*$ -products are equivalent iff they have the same characteristic class.*

4 Vey- $*$ products

Let (M, ω) be a symplectic manifold, and ∇ a torsion free symplectic connection with covariant derivative ∂ . Define $\partial u = du$, $\partial^2 u = \partial(\partial u)$, $\partial^k u = \partial(\partial^{k-1} u)$ and so on. It is simple to see that $\partial^k u$ is a symmetric contravariant k -tensor. Let π be the Poisson bivector field on M . By π^k , we denote the $2k$ -tensor:

$$(\pi^k)^{i_1 \dots i_k, j_1 \dots j_k} = \pi^{i_1 j_1} \pi^{i_2 j_2} \dots \pi^{i_k j_k}.$$

Set

$$P_{\nabla}^k(u, v) = \langle \pi^k, \partial^k u \otimes \partial^k v \rangle = \pi^{i_1 j_1} \dots \pi^{i_k j_k} (\partial_{i_1} \dots \partial_{i_k} u) (\partial_{j_1} \dots \partial_{j_k} v).$$

In particular, $P_{\nabla}^0(u, v) = uv$ and $P_{\nabla}^1(u, v) = \{u, v\}$.

Definition 4.1 [5] *A Vey $*$ -product is a star product on $C^\infty(M)[[\hbar]]$ such that*

$$u *_{\hbar} v = \sum \frac{1}{k!} \left(-\frac{i\hbar}{2}\right)^k Q_k(u, v), \quad (21)$$

where Q_k is a bidifferential operator of maximum order k in each argument and its principal symbol coincides with that of $P_{\nabla}^k(u, v)$.

The main result of the section is

Theorem 4.2 *Any Fedosov $*$ -product is a Vey $*$ -product. Moreover, if $D = -\delta + \partial + [\frac{i}{\hbar}\gamma, \cdot]$ as in Theorem 2.2, then*

$$Q_2(u, v) = P_{\nabla}^2(u, v) + C_2(u, v), \quad (22)$$

where C_2 is a bidifferential operator of maximum order 1 in each argument.

Given a Fedosov connection $D = -\delta + \partial + [\frac{i}{\hbar}\gamma, \cdot]$, any element

$$a(x, y, \hbar) = \sum \hbar^i a_{i, j_1 \dots j_k}(x) y^{j_1} \dots y^{j_k} \quad (23)$$

in W_D is determined by the iteration formula (18). Assume that $a_0(x) = \sigma(a) = a(x, 0, \hbar) \in C^\infty(M)$. It is clear that each coefficient in Equation (23) can be expressed as $a_{i, j_1 \dots j_k}(x) = \mathcal{D}_{i, j_1 \dots j_k} a_0$ for certain differential operator $\mathcal{D}_{i, j_1 \dots j_k}$. We will say that the term $\hbar^i a_{i, j_1 \dots j_k}(x) y^{j_1} \dots y^{j_k}$ is of degree $(i + k, s)$ if $\mathcal{D}_{i, j_1 \dots j_k}$ is a differential operator of degree s .

Proposition 4.3 *Under the same hypothesis as above, assume that*

$$a(x, y, \hbar) = \sum \hbar^i a_{i,j_1 \dots j_k}(x) y^{j_1} \dots y^{j_k} \in W_D.$$

Then

(i). $a_{i,j_1 \dots j_k}(x) = \mathcal{D}_{i,j_1 \dots j_k} a_0$, where $\mathcal{D}_{i,j_1 \dots j_k}$ is a differential operator of degree not greater than $i + k$.

(ii).

$$a(x, y, 0) = \sum_{k=0}^{\infty} \frac{1}{k!} (\partial_{i_1} \dots \partial_{i_k} a_0) y^{i_1} \dots y^{i_k} + H,$$

where all terms in the remainder H are of degree (l, k) with $l < k$.

(iii). *For any j , the order of the differential operator $D_{1,j}$ is not greater than 1.*

Proof. Since $a(x, y, \hbar)$ is generated by iteration formula (18), we prove (i) by induction. For this purpose, we need to analyse the effect of the operators $\delta^{-1}\partial$ and $\delta^{-1}[\frac{i}{\hbar}\gamma, \cdot]$ on any term of degree $(i + k, s)$. It is obvious that $\delta^{-1}\partial$ maps terms of degree $(i + k, s)$ to those of degree $(i + k + 1, s + 1)$. On the other hand, it is not difficult to check that $\delta^{-1}[\frac{i}{\hbar}\gamma, \cdot]$ maps terms of degree $(i + k, s)$ to those of degree (m, s) with $m \geq i + k$. Claim (i) thus follows immediately.

To prove (ii), we need to analyse the \hbar^0 -terms produced from $\delta^{-1}[\frac{i}{\hbar}\gamma, \cdot]$ in iteration formula (18). The only possibility would be $\delta^{-1}(-i\hbar\{\frac{i}{\hbar}\gamma_0, \tilde{a}\})$, where $\gamma_0 = \gamma(x, y, 0, dx)$ and $\tilde{a} = a_{0,j_1 \dots j_k}(x) y^{j_1} \dots y^{j_k}$ is assumed of degree (l, k) . By Part (i), we know that $l \leq k$. Now $\delta^{-1}(-i\hbar\{\frac{i}{\hbar}\gamma_0, \tilde{a}\}) = \delta^{-1}\{\gamma_0, \tilde{a}\}$. Since γ_0 is at least cubic in y ($\deg \gamma \geq 3$), the latter is of degree (l, k') with k' being at least $k + 2$. Therefore, all terms of degree (k, k) in $a(x, y, 0)$ come solely from the iteration of the operator $\delta^{-1}\partial$ on elements of the same form. The conclusion thus follows immediately.

To prove (iii), we will concentrate on those terms of degree $(l, 2)$:

$$\sum \hbar a_{1,j}(x) y^j. \tag{24}$$

Since the operator $\delta^{-1}\partial$ maps any (t, s) -term into that of degree $(t + 1, s + 1)$, and $\delta^{-1}\partial$ does not produce any \hbar , the only possible way to obtain such a term is when $\delta^{-1}\partial$ acts on those of the form $\hbar a_1(x)$. This however will never happen since we already assume that $a(x, 0, \hbar) = a_0(x)$ is independent of \hbar .

As for the operator $\delta^{-1}[\frac{i}{\hbar}\gamma, \cdot]$, the only possible way of generating a term of the form (24) is $\delta^{-1}(-i\hbar\{\frac{i}{\hbar}\gamma', a'\}) = \delta^{-1}\{\gamma', a'\}$, where γ' is the term in γ having the form: $\hbar\gamma_{i,j}(x) y^i dx^j$ and a' is the term in $a(x, y, \hbar)$ of the form: $a' = a_{0,m}(x) y^m$. Thus, $\delta^{-1}\{\gamma', a'\} = \hbar\gamma_{ij}(x) a_{0,m}(x) \omega^{im}(x) y^j$ is of degree $(1, 2)$. This concludes the proof.

□

An immediate consequence of Proposition 4.3 is the following:

Corollary 4.4 *Let $a(x, y, \hbar) \in W_D$, and let $x_0 \in M$ be any point. Then, $a(x_0, y, \hbar) = 0, \forall y \in T_{x_0}M$ iff $\text{jet}_{x_0}^\infty a(x, \hbar) = 0$, where $a(x, \hbar) = a(x, 0, \hbar) = \sigma(a(x, y, \hbar))$.*

Proof. One direction follows directly from Part (1) of Proposition 4.3. To prove the other direction, let $a_0(x) = a(x, 0, 0)$ and assume that $a(x, \hbar) = a_0(x) + \hbar a_1(x) + \hbar^2 a_2(x) + \dots$. We shall prove, as the first step, that $\text{jet}_{x_0}^\infty a_0(x) = 0$.

Clearly, $a_0(x_0) = 0$. Assume that all derivatives of a_0 up to the k -th order vanish at x_0 . According to Part (ii) of Proposition 4.3,

$$a(x_0, y, 0) = \sum_{k=0}^{\infty} \frac{1}{k!} (\partial_{i_1} \dots \partial_{i_k} a_0) y^{i_1} \dots y^{i_k} + H,$$

where all terms in H are of degree (l, k) with $l < k$. Since the coefficient of $y^{i_1} \dots y^{i_k}$ is zero, it follows that $\frac{1}{k!} (\partial_{i_1} \dots \partial_{i_k} a_0) + (\mathcal{D}a_0)(x_0) = 0$, for some differential operator \mathcal{D} of degree less than k . By using the induction assumption, we deduce that $(\partial_{i_1} \dots \partial_{i_k} a_0)(x_0) = 0$. This proves that $\text{jet}_{x_0}^\infty a_0(x) = 0$. Let $a_0(x, y, \hbar) \in W_D$ be the parallel section corresponding to $a_0(x)$. Then $a_0(x_0, y, \hbar) = 0, \forall y \in T_{x_0}M$. By considering the element $\frac{1}{\hbar}[a(x, y, \hbar) - a_0(x_0, y, \hbar)] \in W_D$, one yields that $\text{jet}_{x_0}^\infty a_1(x) = 0$. The conclusion thus follows by using this argument repeatedly. \square

Proof of Theorem 4.2 Let $a_0(x)$ and $b_0(x)$ be any functions on M , and $a(x, y, \hbar)$ and $b(x, y, \hbar)$ their corresponding parallel sections in W_D . Assume that

$$a(x, y, \hbar) = \sum \hbar^i a_{i, j_1 \dots j_k}(x) y^{j_1} \dots y^{j_k}, \quad \text{and}$$

$$b(x, y, \hbar) = \sum \hbar^i b_{i, j_1 \dots j_k}(x) y^{j_1} \dots y^{j_k}.$$

By the definition of Moyal-Weyl product,

$$\begin{aligned} & a(x, y, \hbar) * b(x, y, \hbar)|_{y=0} \\ &= \sum \left(-\frac{i}{2}\right)^p \frac{1}{p!} \hbar^{k+l+p} a_{k, i_1 \dots i_p}(x) b_{l, j_1 \dots j_p}(x) \pi^{t_1 s_1} \dots \pi^{t_p s_p} \frac{\partial(y^{i_1} \dots y^{i_p})}{\partial y^{t_1} \dots \partial y^{t_p}} \frac{\partial(y^{j_1} \dots y^{j_p})}{\partial y^{s_1} \dots \partial y^{s_p}} \\ &= \sum_n \sum_{k+l+p=n} \left(-\frac{i}{2}\right)^p \frac{1}{p!} \hbar^{k+l+p} a_{k, i_1 \dots i_p}(x) b_{l, j_1 \dots j_p}(x) \pi^{t_1 s_1} \dots \pi^{t_p s_p} \frac{\partial(y^{i_1} \dots y^{i_p})}{\partial y^{t_1} \dots \partial y^{t_p}} \frac{\partial(y^{j_1} \dots y^{j_p})}{\partial y^{s_1} \dots \partial y^{s_p}} \\ &= \sum_n \sum_{k+l+p=n} \left(-\frac{i}{2}\right)^p \frac{1}{p!} \hbar^{k+l+p} (\mathcal{D}_{k, i_1 \dots i_p} a_0)(\mathcal{D}_{l, j_1 \dots j_p} b_0) \pi^{t_1 s_1} \dots \pi^{t_p s_p} \frac{\partial(y^{i_1} \dots y^{i_p})}{\partial y^{t_1} \dots \partial y^{t_p}} \frac{\partial(y^{j_1} \dots y^{j_p})}{\partial y^{s_1} \dots \partial y^{s_p}} \end{aligned}$$

According to Proposition 4.3, the order of the differential operator $\mathcal{D}_{k, i_1 \dots i_p}$ is not greater than $k+p$, which is less than or equal to n , similarly for the order of $\mathcal{D}_{l, j_1 \dots j_p}$. If $l \neq 0$, then $k+p = n-l < n$. The order of $\mathcal{D}_{k, i_1 \dots i_p}$ is less than n . Similarly if $k \neq 0$, the order of $\mathcal{D}_{l, j_1 \dots j_p}$ is less than n . So in order to have the maximum order in both arguments, it is necessary that $k = l = 0$, and in this case $p = n$. Using Part (ii) of Proposition 4.3, it is simple to see that the principal part is

$$\sum (-\frac{i\hbar}{2})^n \frac{1}{n!} (\partial_{i_1} \cdots \partial_{i_n} a_0) (\partial_{j_1} \cdots \partial_{j_n} b_0) \pi^{i_1 j_1} \cdots \pi^{i_n j_n}.$$

For the \hbar^2 -term, we need $k + l + p = 2$. If $k = l = 0$ and $p = 2$, we obtain the principal term. When $k = p = 1$ and $l = 0$, then $a_{1,i_1}(x) = \mathcal{D}_{1,i_1} a_0$ and $b_{0,j_1}(x) = \mathcal{D}_{0,j_1} b_0$. According to Proposition 4.3, both \mathcal{D}_{1,i_1} and \mathcal{D}_{0,j_1} have degrees not greater than 1; similarly for the case that $l = p = 1$ and $k = 0$.

This concludes the proof. □

As an immediate consequence, we obtain the following result, which was first proved by Lichnerowicz using homological methods [18].

Corollary 4.5 (*Lichnerowicz*) *Any \ast -product on a symplectic manifold is equivalent to a Vey- \ast -product.*

Remark As we know, the equivalence class of a Fedosov \ast -product is determined by the class of its Weyl curvature in $H^2(M)[[\hbar]]$, which is independent of the symplectic connection in the construction. On the other hand, as we see in Theorem 4.2, the symplectic connection is indeed completely reflected in the Fedosov \ast -product itself. In fact if two Fedosov connections $D_i = -\delta + \partial_i + [\frac{i}{\hbar}\gamma_i, \cdot]$ induce an identical \ast -product on $C^\infty(M)[[\hbar]]$, it is necessary that their symplectic connections coincide, i.e., $\partial_1 = \partial_2$. We would like to conjecture that:

Two Fedosov connections $D_i = -\delta + \partial_i + [\frac{i}{\hbar}\gamma_i, \cdot]$ induce an identical \ast -product on $C^\infty(M)[[\hbar]]$, iff $D_1 = D_2$.

To prove this, one needs to prove that the Weyl curvatures of D_1 and D_2 are not only cohomologous, but indeed coincide, or equivalently $W_{D_1} = W_{D_2}$. In other words, one needs to decode the Weyl curvature (not only its class in $H^2(M)[\hbar]$) from a Fedosov \ast -product directly.

We end this section with the following inverse question:

Question Is every Vey \ast -product necessary a Fedosov \ast -product?

5 Quantization of lagrangian submanifolds

Lagrangian submanifolds play a fundamental role in the study of symplectic manifolds. In fact, according to the “symplectic creed”, everything can be thought of as a lagrangian submanifold [23]. It is very natural to ask: what should lagrangian submanifolds correspond to under a deformation quantization? This is what we aim to answer in this section.

Lemma 5.1 *Let \ast_\hbar be the Moyal-Weyl product for the symplectic vector space $V \cong \mathbb{R}^{2n}$ with the standard symplectic structure. Suppose that $L \subset V$ is a lagrangian subspace. If both f and $g \in C^\infty(M)$ vanish on L , so does $f \ast_\hbar g$.*

Proof. Note that the Moyal-Weyl formula is independent of the choice of linear coordinates. Let us choose a lagrangian subspace K supplementary to L , and linear coordinates $(q_1, \dots, q_n) \in L$, $(p_1, \dots, p_n) \in K$ such that $\omega(q_i, p_j) = \delta_{ij}$. Then the claim follows directly from Equation (2). \square

Let L be a lagrangian submanifold. For any $x \in L$, by W_x^L we denote the subspace of the Weyl algebra W_x consisting of all elements which vanish when being restricted to $T_x L$. According to Lemma 5.1, W_x^L is a subalgebra. Let $W^L = \cup_{x \in L} W_x^L$. Then W^L is a subbundle of the Weyl bundle. Similarly, we can define W_1^L , W_2^L , etc. according to the natural filtration in W .

We say an element $a \in W_p \otimes \Lambda^q$ is in the subspace $(W_p \otimes \Lambda^q)_L$ if for any $v_1, \dots, v_q \in T_x L$, $a(v_1, \dots, v_q) \in W_x^L$. By $(W \otimes \Lambda)_L$, we denote the direct sum $\oplus_{p,q} (W_p \otimes \Lambda^q)_L$. It is clear that both δ and δ^{-1} preserve the space $(W \otimes \Lambda)_L$. More precisely, δ maps $(W_p \otimes \Lambda^q)_L$ into $(W_{p-1} \otimes \Lambda^{q+1})_L$, while δ^{-1} maps $(W_p \otimes \Lambda^q)_L$ into $(W_{p+1} \otimes \Lambda^{q-1})_L$.

Let ∇ be a torsion-free symplectic connection on M such that L is totally geodesic. Then its induced differential ∂ maps $(W \otimes \Lambda)_L$ into $(W \otimes \Lambda^{q+1})_L$.

Proposition 5.2 *As in Theorem 2.2, if in addition, L is totally geodesic with respect to the symplectic connection ∇ , and is lagrangian with respect to the Weyl curvature Ω , i.e., $i^* \Omega = 0$, where $i : L \rightarrow M$ is the embedding, then γ belongs to $(W_3 \otimes \Lambda^1)_L$. Therefore, the corresponding Fedosov connection D preserves $(W \otimes \Lambda)_L$.*

Proof. According to Equation (17), γ is determined by the iteration formula

$$\gamma = \delta^{-1} \tilde{\Omega} + \delta^{-1} (\partial \gamma + \frac{i}{\hbar} \gamma^2), \quad (25)$$

where $\tilde{\Omega} = \Omega - \omega + R$. It is easy to see that $R \in (W_2 \otimes \Lambda^2)_L$ since L is a totally geodesic lagrangian submanifold. Therefore, it follows that $\tilde{\Omega} \in (W_2 \otimes \Lambda^2)_L$. It is then clear, from the above iteration formula, that $\gamma \in (W_3 \otimes \Lambda^1)_L$ since $(W \otimes \Lambda^*)_L$ is preserved under all the operations involved in Equation (25). \square

Let W_D^L denote the subspace of W_D consisting of sections whose restriction to L belong to W^L .

As an immediate consequence, we have

Corollary 5.3 *Under the same hypothesis as in Proposition 5.2, W_D^L is a subalgebra of W_D .*

By $C_L^\infty(M)$, we denote the space of smooth functions on M which vanish on L .

Proposition 5.4 *Under the same hypothesis as in Proposition 5.2,*

$$\sigma(W_D^L) = C_L^\infty(M)[[\hbar]],$$

where σ is the isomorphism between W_D and $C^\infty(M)[[\hbar]]$ as introduced in Section 2.

Therefore, $C_L^\infty(M)[[\hbar]]$ is a subalgebra of the Fedosov $*$ -algebra $(C^\infty(M)[[\hbar]], *_\hbar)$.

Proof. According to Theorem 2.3, if $a \in W_D$ with $\sigma(a) = a_0$, a is determined by the iteration formula

$$a = a_0 + \delta^{-1}(\partial a + [\frac{i}{\hbar}\gamma, a]).$$

If $a_0 \in C_L^\infty(M)[[\hbar]]$, which means that $a_0 \in W^L$, it follows immediately that $a \in W^L$. This is because $\gamma \in (W_3 \otimes \Lambda^1)_L$, and all the operators involved preserve the space $(W \otimes \Lambda)_L$.

Conversely, if $a \in W_D^L$, it is obvious that $a_0 = \sigma(a) \in W^L$.

□

Example 5.5 If $f : M \longrightarrow N$ is a symplectic diffeomorphism, its graph $G_f = \{(x, f(x)) | x \in M\}$ is a lagrangian submanifold of $S = M \times \bar{N}$. Moreover if $f^*\Omega_N = \Omega_M$, where Ω_M and Ω_N are Weyl curvatures on M and N , respectively, G_f is lagrangian with respect to $(\Omega_M, \bar{\Omega}_N)$. Let S be equipped with a product symplectic connection $\nabla \times \tilde{\nabla}$. It is easy to see that G_f is totally geodesic iff $\tilde{\nabla} = f_*\nabla$. In this case, $C_L^\infty(S)$ is a subalgebra of the corresponding Fedosov $*$ -product $(C^\infty(S)[[\hbar]], *_\hbar)$. This implies that f_* is an algebra morphism between Fedosov $*$ -algebras $(C^\infty(M)[[\hbar]], *_\hbar)$ and $(C^\infty(N)[[\hbar]], *_\hbar)$.

The following well known lemma indicates that a totally geodesic symplectic connection always exists for any given lagrangian submanifold.

Lemma 5.6 Given any lagrangian submanifold $L \subset M$, there always exists a torsion-free symplectic connection on M such that L is totally geodesic.

Proof. First, take any torsion-free connection ∇ on M such that L is totally geodesic. Any other connection can be written as

$$\tilde{\nabla}_X Y = \nabla_X Y + S(X, Y), \quad \forall X, Y \in \mathcal{X}(M), \quad (26)$$

where S is a $(2, 1)$ -tensor. Clearly, $\tilde{\nabla}$ is torsion-free iff S is symmetric, i.e., $S(X, Y) = S(Y, X)$ for any $X, Y \in \mathcal{X}(M)$.

$\tilde{\nabla}$ is symplectic iff $\tilde{\nabla}_X \omega = 0$. The latter is equivalent to

$$\omega(S(X, Y), Z) - \omega(S(X, Z), Y) = (\nabla_X \omega)(Y, Z). \quad (27)$$

Let S be the $(2, 1)$ -tensor defined by the equation:

$$\omega(S(X, Y), Z) = \frac{1}{3}[(\nabla_X \omega)(Y, Z) + (\nabla_Y \omega)(X, Z)]. \quad (28)$$

Clearly, $S(X, Y)$, defined in this way, is symmetric with respect to X and Y . Now

$$\begin{aligned}
& \omega(S(X, Y), Z) - \omega(S(X, Z), Y) \\
&= \frac{1}{3}[(\nabla_X \omega)(Y, Z) + (\nabla_Y \omega)(X, Z)] - \frac{1}{3}[(\nabla_X \omega)(Z, Y) + (\nabla_Z \omega)(X, Y)] \\
&= \frac{1}{3}[(\nabla_X \omega)(Y, Z) + (\nabla_Y \omega)(X, Z) + (\nabla_X \omega)(Y, Z) + (\nabla_Z \omega)(Y, X)] \\
&= (\nabla_X \omega)(Y, Z),
\end{aligned}$$

where the last step follows from the identity:

$$(\nabla_X \omega)(Y, Z) + (\nabla_Y \omega)(Z, X) + (\nabla_Z \omega)(X, Y) = 0.$$

This means that $\tilde{\nabla}$ is a torsion-free symplectic connection.

From Equation (28), it follows that if X, Y and Z are all tangent to L , $\omega(S(X, Y), Z) = 0$ since L is totally geodesic with respect to ∇ . Hence, $S(X, Y)$ is tangent to L whenever X, Y are tangent to L . In other words, L is totally geodesic with respect to $\tilde{\nabla}$.

□

Since a $*$ -product is always equivalent to a Fedosov $*$ -product, as a consequence, any lagrangian submanifold, under a deformation quantization, becomes a subalgebra after some “quantum correction”. More precisely, we have

Theorem 5.7 *Let $*_{\hbar}$ be a $*$ -product on a symplectic manifold (M, ω) with characteristic class $[\Omega] \in H^2(M)[[\hbar]]$. Suppose that L is a lagrangian submanifold such that $i^*[\Omega] \in H^2(L)[[\hbar]]$ vanishes, where $i : L \rightarrow M$ is the embedding. Then there exists an operator: $T_{\hbar} = 1 + \hbar T_1 + \hbar^2 T_2 + \dots$, with T_i being differential operators on M , such that $T_{\hbar}(C_L^\infty(M)[[\hbar]])$ is a subalgebra of $(C^\infty(M)[[\hbar]], *_{\hbar})$.*

Proof. By assumption, $i^*\Omega$ is an exact two-form on L , i.e., $i^*\Omega = d\theta_L$ for some $\theta_L \in \Omega^1(L)$. Extending θ_L to a one-form on M , we may assume that $i^*\Omega = di^*\theta$ for some $\theta \in \Omega^1(M)$. Let $\tilde{\Omega} = \Omega - d\theta$, and take a torsion-free symplectic connection ∇ such that L is totally geodesic. Let $\bar{*}_{\hbar}$ be the corresponding Fedosov $*$ -product with Weyl curvature $\tilde{\Omega}$. Then $C_L^\infty(M)[[\hbar]]$ is a $\bar{*}_{\hbar}$ -subalgebra. According to Theorem 3.10, $*_{\hbar}$ and $\bar{*}_{\hbar}$ are equivalent $*$ -products. The conclusion thus follows immediately.

□

Remark (1). The quantum counterparts of lagrangian submanifolds, according to Lu [19], are left ideals. However, it is not clear how this can be realized for $*$ -algebras in our case. It seems that a possible candidate would be the space $C_L^\infty(M)[[\hbar]]$ modified in a certain way.

(2). For a symplectic manifold M , a coisotropic submanifold is a submanifold C such that the space of functions vanishing on C becomes a Poisson subalgebra. It is natural to expect that the above result can be generalized to any coisotropic submanifolds. But we cannot prove this at the moment because it is not clear if there always exists a symplectic connection such that C is totally geodesic for a general coisotropic submanifold C .

6 Quantum momentum maps

This section is devoted to the study of deformation quantization of a symplectic G -space. In particular, we will introduce the notion of quantum momentum maps, which plays the role of a quantum analogue of the usual momentum maps.

Let (M, ω) be a symplectic G -space with action $\Phi_g : M \longrightarrow M, \forall g \in G$. A $*$ -product on M is called G -equivariant if for any u and $v \in C^\infty(M)[[\hbar]]$,

$$\Phi_g^*(u *_\hbar v) = (\Phi_g^* u) *_\hbar (\Phi_g^* v). \quad (29)$$

In general, M does not necessary admit a G -equivariant $*$ -product. It is known [5] [18] that the existence of such a $*$ -product is closely related to the existence of a G -invariant connection on the manifold. More precisely,

Proposition 6.1 *Let M be a symplectic G -space. M admits a G -equivariant natural $*$ -product iff there exists a G -invariant connection on M , where by a natural $*$ -product, we mean a $*$ -product:*

$$u *_\hbar v = \sum_k \hbar^k C_k(u, v),$$

where the \hbar^2 -term C_2 is a bidifferential operator of order 2 in each argument.

Proof. Assume that there exists a G -equivariant natural $*$ -product:

$$u *_\hbar v = uv - \frac{i\hbar}{2} \{u, v\} + \frac{1}{2} \left(\frac{i\hbar}{2} \right)^2 Q_2(u, v) + \dots$$

According to Proposition 10.1, there is a unique symplectic connection ∇ such that

$$Q_2(u, v) = P_{\nabla}^2(u, v) + H(u, v),$$

where $H(u, v)$ is a bidifferential operator of maximum order 1 in each argument. It is then clear that ∇ is G -invariant.

Conversely, suppose that there exists a G -invariant connection on M . Using the standard method, we can always make it into a G -invariant torsion free symplectic connection ∇ . Then, the corresponding Fedosov $*$ -product (with symplectic connection ∇ , and Weyl curvature ω) will be G -equivariant.

□

A G -invariant connection always exists if G is compact. However, when G is non-compact, there exist some cases where G -invariant connections do not exist. Various attempts have been made in order to deal with such a situation. For details, readers can consult [2] [3] [14].

In what follows, nevertheless, we will always assume that $*_\hbar$ is a G -equivariant $*$ -product. Then, the corresponding infinitesimal action $\xi \longrightarrow \hat{\xi}$ defines a Lie algebra homomorphism from \mathfrak{g} to the derivation space $Der C^\infty(M)[[\hbar]]$ of the $*$ -algebra.

By $U\mathfrak{g}[[\hbar]]$, we denote the space of formal power series of \hbar with coefficients in the universal enveloping algebra $U\mathfrak{g}$. Let \mathfrak{g}_\hbar be the deformed Lie algebra: $\mathfrak{g}_\hbar = \mathfrak{g}[[\hbar]]$ with the bracket being defined as

$$[X, Y]_\hbar = -i\hbar[X, Y], \quad \forall X, Y \in \mathfrak{g}[[\hbar]]. \quad (30)$$

Then $U\mathfrak{g}[[\hbar]]$ can be identified with the universal enveloping algebra of \mathfrak{g}_\hbar , and therefore inherits an associative algebra structure.

Definition 6.2 *A quantum momentum map is a homomorphism of associative algebras:*

$$\mu_\hbar : U\mathfrak{g}[[\hbar]] \longrightarrow C^\infty(M)[[\hbar]],$$

such that for any $\xi \in \mathfrak{g}$,

$$\hat{\xi} = \text{ad} \frac{i}{\hbar} \mu_\hbar(\xi), \quad (31)$$

where both sides are considered as derivations on $C^\infty(M)[[\hbar]]$.

It is obvious, from definition, that a necessary condition for the existence of a momentum map is that the derivation $\rho_\xi f = \hat{\xi}f$, $\forall \xi \in \mathfrak{g}$ be inner. Let us first assume that this is true. Thus there is a linear map from \mathfrak{g} to $C^\infty(M)[[\hbar]]$, denoted by $\xi \longrightarrow a_\xi$, such that for any $f \in C^\infty(M)$ and $\xi \in \mathfrak{g}$,

$$\hat{\xi}f = [\frac{i}{\hbar} a_\xi, f]. \quad (32)$$

Therefore,

$$\begin{aligned} \widehat{[\xi, \eta]}f &= [\hat{\xi}, \hat{\eta}]f \\ &= [\frac{i}{\hbar} a_\xi, [\frac{i}{\hbar} a_\eta, f]] - [\frac{i}{\hbar} a_\eta, [\frac{i}{\hbar} a_\xi, f]] \\ &= [[\frac{i}{\hbar} a_\xi, \frac{i}{\hbar} a_\eta], f]. \end{aligned}$$

On the other hand, by definition,

$$\widehat{[\xi, \eta]}f = [\frac{i}{\hbar} a_{[\xi, \eta]}, f].$$

Therefore,

$$[a_{[\xi, \eta]} - \frac{i}{\hbar} [a_\xi, a_\eta], f] = 0, \quad \forall f \in C^\infty(M).$$

So $a_{[\xi, \eta]} - \frac{i}{\hbar} [a_\xi, a_\eta]$, as a function in $C^\infty(M)[[\hbar]]$, is constant.

Define $\lambda : \wedge^2 \mathfrak{g} \longrightarrow \mathbb{C}[[\hbar]]$ by

$$\lambda(\xi, \eta) = a_{[\xi, \eta]} - \frac{i}{\hbar} [a_\xi, a_\eta], \quad \forall \xi, \eta \in \mathfrak{g}. \quad (33)$$

Proposition 6.3 (i). λ is a Lie algebra 2-cocycle;

- (ii). *its cohomology class $[\lambda] \in H^2(\mathfrak{g}, \mathbb{C}[[\hbar]]) \cong H^2(\mathfrak{g}) \otimes \mathbb{C}[[\hbar]]$ is independent of the choice of the linear map a_ξ ;*
- (iii). *quantum momentum map exists iff $[\lambda] = 0$.*

Proof. Assertions (i)-(ii) are quite obvious, and left for the reader to check.

For (iii), suppose that a quantum momentum map μ_\hbar exists. Then we may take $a_\xi = \mu_\hbar(\xi)$ as our linear map. In this case,

$$\begin{aligned} \lambda(\xi, \eta) &= a_{[\xi, \eta]} - \frac{i}{\hbar} [a_\xi, a_\eta] \\ &= \mu_{\frac{i}{\hbar}}[\xi, \eta] - \frac{i}{\hbar} [\mu_\hbar \xi, \mu_\hbar \eta] \\ &= \mu_{\frac{i}{\hbar}}([\xi, \eta] - \frac{i}{\hbar} [\xi, \eta]_\hbar) \\ &= 0. \end{aligned}$$

Conversely, if $[\lambda] = 0$, by adding a suitable coboundary we can always choose a linear map $a : \mathfrak{g} \longrightarrow C^\infty(M)[[\hbar]]$ such that Equation (32) holds and $a_{[\xi, \eta]} - \frac{i}{\hbar} [a_\xi, a_\eta] = 0$. In other words, a is a Lie algebra homomorphism from \mathfrak{g}_\hbar to the commutator Lie algebra of $C^\infty(M)[[\hbar]]$. Therefore, it extends to an associative algebra morphism:

$$\mu_\hbar : U\mathfrak{g}_\hbar \longrightarrow (C^\infty(M)[[\hbar]], *_\hbar).$$

This concludes the proof. □

According to Theorem 8.2, derivations are automatically inner if $H^1(M) = 0$. Thus we have

Theorem 6.4 *There exists a quantum momentum map if $H^1(M) = 0$ and $H^2(\mathfrak{g}) = 0$. In particular, a quantum momentum map exists if M is simply connected and \mathfrak{g} is semi-simple.*

We note that the above condition is exactly the same sufficient condition for the existence of a classical momentum map [1]. However, there are many cases where classical momentum maps still exist even if this condition is no longer satisfied. It is reasonable to expect that this phenomenon would happen for quantum momentum maps as well. However, we do not know too many examples except for the following:

Example 6.5 *Suppose that Q is a G -manifold with action φ_g , which admits a G -invariant torsion free connection ∇ . Let $M = T^*Q$ be equipped with the standard cotangent bundle symplectic structure. The G -action naturally lifts to a symplectic action Φ_g on $M = T^*Q$ with an equivariant momentum map $J : T^*Q \longrightarrow \mathfrak{g}^*[1]$:*

$$\langle J(\xi_q), X \rangle = \langle \hat{X}, \xi_q \rangle, \quad \xi_q \in T_q Q, \forall X \in \mathfrak{g},$$

where \hat{X} is the vector field on Q generated by $X \in \mathfrak{g}$.

To any differential operator D , we assign its (complete) symbol as the polynomial S_D on T^*Q given by

$$S_D(\xi_q) = De^{\langle \exp_q^{-1}x, \xi_q \rangle}|_{x=q}, \quad \forall \xi_q \in T_qQ, \quad (34)$$

where $\exp_q : T_qQ \rightarrow Q$ is the usual exponential map, defined in a neighborhood of 0, corresponding to the connection ∇ .

This assignment in fact establishes an isomorphism between the space \mathcal{D} of differential operators and that of polynomials on T^*Q .

Deform the Lie bracket structure on $\mathcal{X}(Q)$ according to:

$$\begin{aligned} Z_{\hbar}f &= -i\hbar Zf, \quad \forall Z \in \mathcal{X}(Q) \text{ and } f \in C^\infty(Q); \text{ and} \\ [Y, Z]_{\hbar} &= -i\hbar[Y, Z], \quad \forall Y, Z \in \mathcal{X}(Q). \end{aligned}$$

Since differential operators are generated by $C^\infty(Q)$ and $\mathcal{X}(Q)$ over the module $C^\infty(Q)$, this deformed bracket induces an \hbar -depending multiplication on \mathcal{D}^1 , which in turn induces a \hbar -depending multiplication on the space of polynomials on T^*Q , hence a $*$ -product on T^*Q . It is simple to see that for any $D \in \mathcal{D}$,

$$\Phi_g^* S_D = S_{g^{-1} \cdot D},$$

where $g \cdot D \stackrel{\text{def}}{=} \varphi_{g^{-1}}^* \circ D \circ \varphi_g^*$. In the equation above, by letting $g = \exp tX$, $\forall X \in \mathfrak{g}$, and taking derivative at $t = 0$, one obtains immediately that

$$\hat{X}(S_D) = S_{[\hat{X}, D]},$$

where \hat{X} on the LHS refers to the vector field on Q , while \hat{X} on the RHS stands for the one on T^*Q corresponding to $X \in \mathfrak{g}$. This implies that

$$\hat{X}f = \left[\frac{i}{\hbar} J^* l_X, f \right], \quad \forall f \in C^\infty(T^*Q).$$

In other words, $\mu_{\hbar}X = J^* l_X$ defines a quantum momentum map. However, it is not clear how to express $\mu_{\hbar}f$ explicitly for a general $f \in U\mathfrak{g}[[\hbar]]$.

When quantum momentum maps exist, they are, in general, not unique, as in the classical case.

Assume that both μ_{\hbar} and $\nu_{\hbar} : U\mathfrak{g}[[\hbar]] \rightarrow C^\infty(M)[[\hbar]]$ are quantum momentum maps. Let $\tau_{\hbar} : \mathfrak{g} \rightarrow C^\infty(M)[[\hbar]]$ be the map defined as:

$$\tau_{\hbar}(\xi) = \mu_{\hbar}(\xi) - \nu_{\hbar}(\xi), \quad \forall \xi \in \mathfrak{g}.$$

Then, for any $f \in C^\infty(M)$, $\frac{i}{\hbar}[\tau_{\hbar}(\xi), f] = 0$. Thus it follows that $\tau_{\hbar}(\xi)$ is a constant formal polynomial of \hbar . Also, it is easy to see that

$$\tau_{\hbar}([\xi, \eta]) = 0.$$

¹A more intrinsic viewpoint is to think the tangent bundle TQ as a Lie algebroid, and the above construction as deforming the algebroid structure by multiplying the factor $-i\hbar$. Then such a construction admits an immediate generalization to algebroids, which should give rise to a $*$ -product for the Lie-Poisson structure associated with a Lie algebroid.

That is, $\tau_{\hbar} : \mathfrak{g} \longrightarrow \mathbb{C}[[\hbar]]$ is a 1-cocycle. Since all 1-coboundaries are trivial, it follows that the quantum momentum map is unique if $H^1(\mathfrak{g}) = 0$.

Proposition 6.6 *If $H^1(\mathfrak{g}) = 0$, then the quantum momentum map is unique.*

In general, for $\xi \in \mathfrak{g}$, we have $\mu_{\hbar}\xi = \nu_{\hbar}\xi + \tau_{\hbar}\xi$. However, it is not clear how to express $\mu_{\hbar}f$, for a general $f \in U\mathfrak{g}[[\hbar]]$, in terms of ν_{\hbar} and τ_{\hbar} .

To see the relation between a quantum momentum map and a classical one, we start with the following simple

Lemma 6.7 *Suppose that*

$$f *_h g = \sum_k \hbar^k C_k(f, g)$$

is a $$ -product on a symplectic manifold M . Let $X \in \mathcal{X}(M)$ be a vector field on M , which is an inner derivation when being considered as an operator on $C^\infty(M)[[\hbar]]$. Assume that $Xf = [\frac{i}{\hbar}a, f]$, $\forall f \in C^\infty(M)[[\hbar]]$. Then, modulo a constant formal polynomial of \hbar , a is of the form $a = \sum_i \hbar^{2i} a_{2i}$, where a_0 is a hamiltonian function generating the vector field X , and a_{2k} , for $k \geq 1$, is determined by the equation:*

$$\{a_{2k}, f\} = \sum_{i+j=k-1, i \geq 0, j \geq 0} -2 \frac{i}{\hbar} C_{2j+3}(a_{2i}, f).$$

Proof. It is a direct verification, and is left for the reader. □

As a vector space, $U\mathfrak{g}[[\hbar]]$ is canonically isomorphic to $pol(\mathfrak{g}^*)[[\hbar]]$, the space of formal power series of \hbar with coefficients being polynomials on \mathfrak{g}^* . The isomorphism is established by the symmetrization (see [6] for details). Therefore, the algebra structure on $U\mathfrak{g}[[\hbar]]$ induces a $*$ -product on $pol(\mathfrak{g}^*)[[\hbar]]$, which gives rise to a deformation quantization for the Lie-Poisson structure \mathfrak{g}^* . Below, we will identify these two spaces and use them interchangeably if there is no confusion.

Proposition 6.8 *Suppose that $*_{\hbar}$ is a G -equivariant $*$ -product on a symplectic manifold M . Assume that $\mu_{\hbar} : U\mathfrak{g}[[\hbar]] \longrightarrow C^\infty(M)[[\hbar]]$ is a quantum momentum map. Then, M is a hamiltonian G -space, i.e., the symplectic G -action admits an equivariant (classical) momentum map J . Moreover,*

$$\mu_{\hbar}f = J^*f + O(\hbar), \quad \forall f \in pol(\mathfrak{g}^*).$$

Proof. Since $\mu_{\hbar}\xi$, $\forall \xi \in \mathfrak{g}$, depends on \mathfrak{g} linearly, it defines a map $J : X \longrightarrow \mathfrak{g}^*$ uniquely by the relation:

$$\mu_{\hbar}(l_{\xi}) = J^*l_{\xi} + O(\hbar), \quad \forall \xi \in \mathfrak{g}.$$

Then clearly J is a (classical) momentum map according to Lemma 6.7.

Write the 2-cocycle λ defined by Equation (33) as

$$\lambda = \lambda_0 + \hbar \lambda_1 + \cdots.$$

Then each $\lambda_i : \wedge^2 \mathfrak{g} \longrightarrow \mathbb{C}$ is a Lie algebra 2-cocycle. Moreover, it is simple to see that

$$\lambda_0(\xi, \eta) = J^* l_{[\xi, \eta]} - \{J^* l_\xi, J^* l_\eta\}.$$

The vanishing of σ implies that $\sigma_0 = 0$, which means that J is equivariant. The rest of the proposition thus follows trivially.

□

It is, however, not clear whether the converse of Proposition 6.8 is true or not. We end this section by posing the following

QUESTION Does the existence of a classical moment map imply the existence of a quantum moment map?

7 Quantum dual pair

This is a continuation of the last section. We will assume the same hypothesis as in the previous section, and in particular, assume that a quantum momentum map $\mu_\hbar : U\mathfrak{g}[[\hbar]] \longrightarrow C^\infty(M)[[\hbar]]$ exists.

Let $C^\infty(M)^G$ denote the space of all G -invariant functions on M .

Proposition 7.1

$$(\mu_\hbar U\mathfrak{g}[[\hbar]])' \cong C^\infty(M)^G[[\hbar]].$$

Proof. Let $f = \sum_i \hbar^i f_i \in C^\infty(M)[[\hbar]]$. Then f commutes with $\mu_\hbar U\mathfrak{g}[[\hbar]]$ iff it commutes with its generators. That is, $[f, \mu_\hbar \xi] = 0$, $\forall \xi \in \mathfrak{g}$. The latter is equivalent to that $\hat{\xi}f = 0$, or in other words, f is G -invariant.

□

According to this proposition, we have $\mu_\hbar(U\mathfrak{g}[[\hbar]]) \subseteq (C^\infty(M)^G[[\hbar]])'$. In order to describe the commutant $(C^\infty(M)^G[[\hbar]])'$ completely, we need to extend the quantum momentum map μ_\hbar .

It is well-known that the star-product on $\text{pol}(\mathfrak{g}^*)[[\hbar]]$ naturally extends to a star-product on smooth functions $C^\infty(\mathfrak{g}^*)[[\hbar]]$. Below we will indicate that a quantum momentum map, if it exists, extends to $C^\infty(\mathfrak{g}^*)[[\hbar]]$ as well. More precisely, we have

Proposition 7.2 *Let $\mu_\hbar : \text{pol}(\mathfrak{g}^*)[[\hbar]] \longrightarrow C^\infty(M)[[\hbar]]$ be a quantum momentum map. Then it naturally extends to an algebra morphism, denoted by the same notation μ_\hbar , from $C^\infty(\mathfrak{g}^*)[[\hbar]]$ to $C^\infty(M)[[\hbar]]$ such that*

- (i). for any $f \in C^\infty(\mathfrak{g}^*)$, $\mu_\hbar f$ commutes with $C^\infty(M)^G[[\hbar]]$, and
- (ii). for any $f \in C^\infty(\mathfrak{g}^*)$, $\mu_\hbar f = J^* f + O(\hbar)$.

Proof. Given a smooth function $f \in C^\infty(\mathfrak{g}^*)$, for any $x_0 \in M$, let $\tilde{f}_{x_0}(u)$, $u \in \mathfrak{g}^*$ denote its Talyor expansion at the point $u_0 = J(x_0)$. Define²

$$(\mu_\hbar f)(x_0) = (\mu_\hbar \tilde{f}_{x_0})|_{x=x_0}. \quad (35)$$

It is clear that this definition coincides with the original μ_\hbar when f is a polynomial, so it is indeed an extension of the given quantum momentum map. It is also obvious, from definition, that $\mu_\hbar f$ commutes with $C^\infty(M)^G[[\hbar]]$.

It is simple to see that each term in the expansion of $\tilde{f}_{x_0} - f(J(x_0))$, as a function in u , is a homogeneous polynomial in $u - u_0$. Hence, we have $\mu_\hbar f = J^* f + O(\hbar)$.

It remains to check that μ_\hbar is an algebra homomorphism. This follows from the fact that $\widetilde{(f *_\hbar g)}_{x_0} = \tilde{f}_{x_0} *_\hbar \tilde{g}_{x_0}$, for all f and $g \in C^\infty(\mathfrak{g}^*)$.

To see this, we write $f *_\hbar g = \sum \hbar^k C_k(f, g)$. Then, this essentially follows from the fact that $\widetilde{C_k(f, g)}_{x_0} = C_k(\tilde{f}_{x_0}, \tilde{g}_{x_0})$, which is in turn a consequence of the fact that $pol(\mathfrak{g}^*)[[\hbar]]$ is a subalgebra under the $*$ -product on $C^\infty(\mathfrak{g}^*)[[\hbar]]$.

□

Proposition 7.3 *Under the same hypothesis as in Proposition 7.2, if, in addition, the G -action is free and proper,*

$$(C^\infty(M)^G[[\hbar]])' \cong \mu_\hbar(C^\infty(\mathfrak{g}^*)[[\hbar]]).$$

Proof. According to Proposition 7.2, $\mu_\hbar(C^\infty(\mathfrak{g}^*)[[\hbar]]) \subseteq (C^\infty(M)^G[[\hbar]])'$. To prove the other direction, let us assume that $f = \sum_i \hbar^i f_i \in (C^\infty(M)^G[[\hbar]])'$.

It is clear that for any $g \in C^\infty(M)$,

$$\frac{i}{\hbar}[f, g] = \{f_0, g\} + O(\hbar).$$

Then it follows that $\{f_0, g\} = 0$, $\forall g \in C^\infty(M)^G$. This implies that $f_0 = J^* f'_0$ for some smooth function $f'_0 \in C^\infty(\mathfrak{g}^*)$. Now according to Proposition 7.2, $\mu_\hbar f'_0 = J^* f'_0 + O(\hbar) = f_0 + O(\hbar)$. Therefore, $f - \mu_\hbar f'_0 = \hbar \tilde{f}$, where $\tilde{f} \in (C^\infty(M)^G[[\hbar]])'$ since both f and $\mu_\hbar f'_0$ belong to $(C^\infty(M)^G[[\hbar]])'$. By repeating the same argument on \tilde{f} and so on, we deduce that $f \in \mu_\hbar(C^\infty(\mathfrak{g}^*)[[\hbar]])$.

□

Combining Propositions 7.1-7.3, we have

²We are grateful to A. Weinstein for suggesting this method of extension.

Theorem 7.4 *Suppose that $*_{\hbar}$ is a G equivariant $*$ -product on $C^\infty(M)[[\hbar]]$ with a quantum momentum map μ_{\hbar} . Assume that the action is free and proper, then*

$$\begin{aligned} (C^\infty(M)^G[[\hbar]])' &\cong \mu_{\hbar}(C^\infty(\mathfrak{g}^*)[[\hbar]]) \\ (\mu_{\hbar}C^\infty(\mathfrak{g}^*)[[\hbar]])' &\cong C^\infty(M)^G[[\hbar]]. \end{aligned}$$

Recall that if A is an associative algebra and $B \subseteq A$ is a subset, then B' , the commutant of B , is an associative subalgebra of A . If B is also the commutant of B' , then B and B' are called *mutual commutants*.

The notion of mutual commutants is an important concept in the theory of associative algebras, especially in operator algebras. It has been generalized to the context of groups by Roger Howe [16], called dual pairs of groups, in his study of representation theory and mathematical physics. On the classical level, or more precisely on the level of Poisson manifolds, an analogue was introduced by Weinstein [24], which is called dual pair of Poisson manifolds. In fact, Poisson manifolds \mathfrak{g}^* and M/G , together with the Poisson maps $J : M \rightarrow \mathfrak{g}^*$ and $p : M \rightarrow M/G$, consist a dual pair in terms of Weinstein [24]. Now $(C^\infty(\mathfrak{g}^*)[[\hbar]], *_{\hbar})$ provides a deformation quantization for the Lie-Poisson structure \mathfrak{g}^* while $(C^\infty(M)^G[[\hbar]], *_{\hbar})$ quantizes the reduced Poisson space M/G . So Theorem 7.4 equivalently says that under deformation quantization, the classical dual pair \mathfrak{g}^* and M/G becomes mutual commutants. For this reason, we shall call the pair of algebras $(C^\infty(\mathfrak{g}^*)[[\hbar]], *_{\hbar})$ and $(C^\infty(M)^G[[\hbar]], *_{\hbar})$ a *quantum dual pair*. In general, given two Poisson manifolds P_1 and P_2 and their deformation quantization $(C^\infty(P_1)[[\hbar]], *_{\hbar})$ and $(C^\infty(P_2)[[\hbar]], *_{\hbar})$, we say that they consist a *quantum dual pair* if there is a symplectic manifold M and a star-product $(C^\infty(M)[[\hbar]], *_{\hbar})$ on M , and algebra morphisms $\rho_1 : C^\infty(P_1)[[\hbar]] \rightarrow C^\infty(M)[[\hbar]]$ and $\rho_2 : C^\infty(P_2)[[\hbar]] \rightarrow C^\infty(M)[[\hbar]]$ such that $\rho_1(C^\infty(P_1)[[\hbar]])$ and $\rho_2(C^\infty(P_2)[[\hbar]])$ are mutual commutants.

Unfortunately, at the moment we only know a few examples of deformation quantizable Poisson manifolds. In fact, we do not know any other examples of quantum dual pairs besides this and the trivial ones. In particular, it is not clear in general whether a classical dual pair (or Morita equivalent Poisson manifolds [28]) can be quantized to a quantum dual pair or not. The answer to all these questions relies upon how successful it is the deformation quantization theory of Poisson manifolds.

8 Appendix A (Derivations of $*$ -algebras)

In this Appendix, we collect some basic facts concerning derivations of a $*$ -algebra $C^\infty(M)[[\hbar]]$ on a symplectic manifold M . Some of them are well known. However, as we shall see, Fedosov method even sheds new light on understanding these results.

Definition 8.1 *A derivation of a $*$ -algebra $(C^\infty(M)[[\hbar]], *_{\hbar})$ is a formal power series of \hbar with coefficients being linear operators on $C^\infty(M)$: $\delta = D_0 + \hbar D_1 + \cdots + \hbar^i D_i + \cdots$ such that*

$$\delta(f *_{\hbar} g) = \delta f *_{\hbar} g + f *_{\hbar} \delta g, \forall f, g \in C^\infty(M)[[\hbar]]. \quad (36)$$

A derivation is said to be inner if $\delta = \text{ad}_{\frac{i}{\hbar}H} = [\frac{i}{\hbar}H, \cdot]$ for some $H \in C^\infty(M)[[\hbar]]$.

Suppose that $\delta = \sum_i \hbar^i D_i$ is a derivation. By expanding both sides of Equation (36), the \hbar^0 terms yield that $D_0(fg) = D_0(f)g + fD_0(g)$. That is, D_0 is a vector field. By considering the \hbar^1 terms, one obtains that

$$D_0\{f, g\} - \{D_0f, g\} - \{f, D_0g\} = (D_1f)g + f(D_1g) - D_1(fg).$$

Since the LHS is skew-symmetric with respect to f and g while the RHS is symmetric, both terms have to vanish identically. Thus, D_1 is a vector field and D_0 is a symplectic vector field.

Moreover, we have the following (see [5] for an equivalent result, which is however concerning derivations of the corresponding deformed Lie algebra; see also [7]).

Theorem 8.2 *Suppose that $D = D_0 + \hbar D_1 + \cdots \hbar^i D_i + \cdots$ is a derivation of a \ast -algebra $(C^\infty(M)[[\hbar]], \ast_\hbar)$ on a symplectic manifold M . Then,*

- (i). *the operator D_i , for each i , is a differential operator, and in particular D_0 is a symplectic vector field;*
- (ii). *there is a canonical one-one correspondence between derivations and $Z^1(M)[[\hbar]]$;*
- (iii). *under such a correspondence, inner derivations correspond to exact 1-forms in $B^1(M)[[\hbar]]$.*

To begin with, we will consider Fedosov algebras, and assume that $A = W_D$ for some Fedosov connection D . We need a couple of lemmas first.

Lemma 8.3 *Let $K \in \Gamma W$ be a section. Then, $\rho = ad_\hbar^i K = [\frac{i}{\hbar}K, \cdot]$ defines a derivation of W_D iff DK is a scalar closed one-form on M .*

Proof. It is clear that ρ , defined in this way, satisfies the derivation property.

For any $a \in W_D$,

$$D\rho(a) = [\frac{i}{\hbar}DK, a] + [\frac{i}{\hbar}K, Da].$$

If $\rho W_D \subset W_D$, it follows that $[DK, a] = 0$, $\forall a \in W_D$. Thus $DK = \theta$ is a scalar one-form on M , i.e., $\theta \in \Omega^1(M)[[\hbar]]$. Then θ must be closed since $d\theta = D\theta = D^2K = 0$.

The converse follows essentially from the same argument backwards.

□

Lemma 8.4 *For a \ast -product on a symplectic manifold M , any symplectic vector field X may extend to a derivation $\delta = \sum_i \hbar^{2i} D_{2i}$, with D_{2i} being differential operators, such that $D_0 = X$. If X is a hamiltonian vector field, δ may be chosen as an inner derivation.*

Proof. Since every $*$ -product is equivalent to a Fedosov $*$ -product, we may confine ourselves to a Fedosov algebra W_D .

Let $\theta = X \lrcorner \omega$. Then θ is a closed one-form on M . Let $K \in \Gamma(W)$ be the section satisfying

$$DK = \theta \quad \text{and} \quad K|_{y=0} = 0. \quad (37)$$

Note that such a section always exists according to the Fedosov iteration method. In fact, K is uniquely determined by the following iteration formula:

$$K = -\delta^{-1}\theta + \delta^{-1}(\partial K + [\frac{i}{\hbar}\gamma, K]). \quad (38)$$

Take $\delta = ad_{\frac{i}{\hbar}}K$. Then δ is easily seen to be a required derivation. In fact, locally θ is exact, so δ can be expressed as an inner derivation generated by some function on M , which is clearly a formal power series of \hbar^2 with coefficients being differential operators.

If X is a hamiltonian vector field with the hamiltonian function H , we may take $\delta = [\frac{i}{\hbar}H, \cdot]$, which is an inner derivation having the desired property.

□

Proof of Theorem 8.2 According to the observation preceding Theorem 8.2, D_0 is a symplectic vector field. Thus, it extends to a derivation $\delta_0 = D_0 + O(\hbar)$, whose coefficients are differential operators, according to the lemma above. Let $\tilde{\delta} = \frac{1}{\hbar}(\delta - \delta_0)$. Then $\tilde{\delta}$ is a derivation and $\delta = \delta_0 + \hbar\tilde{\delta}$. Applying this process repeatedly, one obtains that $\delta = \sum \hbar^i \delta_i$, where every δ_i is a derivation whose coefficients are differential operators. So δ itself is such a derivation as well.

To continue, without loss of generality, we shall confine ourselves to the case of a Fedosov algebra W_D . Assume that $\delta : W_D \rightarrow W_D$ is a derivation. For any $x_0 \in M$, we define a derivation ρ_{x_0} on W_{x_0} by

$$\rho_{x_0}a(y, \hbar) = (\delta\tilde{a})|_{x=x_0}, \quad \text{where } a(y, \hbar) \in W_{x_0} \text{ and } \tilde{a} \in W_D \text{ such that } \tilde{a}(x_0, y, \hbar) = a(y, \hbar). \quad (39)$$

Clearly, ρ_{x_0} is well defined since δ is a local operator according to Part (i). Since all derivations on W_{x_0} are inner, there is an element $K(x_0) \in W_{x_0}$ such that $\rho_{x_0} = ad_{\frac{i}{\hbar}}K(x_0)$. By requiring that $K(x_0)|_{y=0} = 0$, $K(x_0)$ will be unique. Repeating this process pointwisely, we obtain a global section $K \in \Gamma(W)$ with $K|_{y=0} = 0$ such that

$$\delta\tilde{a} = [\frac{i}{\hbar}K, \tilde{a}], \quad \forall \tilde{a} \in W_D.$$

According to Lemma 8.3, $\theta = DK$ belongs to $Z^1(M)[[\hbar]]$. In this way, we obtain a map φ from the space of derivations $DerC^\infty(M)[[\hbar]]$ to that of closed 1-forms $Z^1(M)[[\hbar]]$.

Conversely, given any $\theta \in Z^1(M)[[\hbar]]$, the equation:

$$DK = \theta, \quad \text{and} \quad K|_{y=0} = 0 \quad (40)$$

always has a unique solution. Thus, $\delta = ad_{\hbar}^i K$ defines a derivation of W_D such that $\varphi\delta = \theta$. In other words, φ is onto. It is also simple to see that φ is injective since the solution to Equation (40) is unique.

Suppose that δ is an inner derivation: $\delta = ad_{\hbar}^i H$ for some $H \in W_D$. Thus, $K = H - H_0$, where $H_0 = H|_{y=0} \in C^\infty(M)[[\hbar]]$, and $\theta = DK = D(H - H_0) = -dH_0$, which is clearly exact.

Conversely, if $\varphi\delta = \theta$ is exact, i.e., $\theta = dH$ for some $H \in C^\infty(M)[[\hbar]]$, we have $D(K - H) = \theta - dH = 0$. That is, $K - H \in W_D$. Thus, $\delta = ad_{\hbar}^i K = ad_{\hbar}^i (K - H)$ is clearly inner. This concludes the proof of the theorem.

□

It is well known that the bracket of any two symplectic vector fields is hamiltonian. As another immediate application of Fedosov method, we obtain the following “quantum” analogue of this fact.

Proposition 8.5 *Let $*_{\hbar}$ be any $*$ -product on a symplectic manifold M . Then the bracket of any derivations is an inner derivation.*

Proof. Without loss of generality, assume that $A = W_D$ for some Fedosov connections, and $\delta_1 = ad_{\hbar}^i K_1$ and $\delta_2 = ad_{\hbar}^i K_2$ are derivations of W_D , where K_1 and K_2 are sections of W . Then, DK_1 and DK_2 belong to $Z^1(M)[[\hbar]]$ according to Lemma 8.3. Now $[\delta_1, \delta_2] = ad_{\hbar}^i(\frac{i}{\hbar}[K_1, K_2])$. It is clear that $K = \frac{i}{\hbar}[K_1, K_2]$ is a section of W and $DK = 0$. Therefore, $[\delta_1, \delta_2]$ is an inner derivation.

□

Remark (1). We note that our definition of inner derivations differs from the usual one by a factor $\frac{i}{\hbar}$. If we modify the usual Hochschild coboundary operator by multiplying the factor $\frac{i}{\hbar}$, Theorem 8.2 implies that the first Hochschild cohomology $H^1(C^\infty(M)[[\hbar]], C^\infty(M)[[\hbar]])$ is isomorphic to $H^1(M)[[\hbar]]$ (see [5]). Similarly, for higher order cohomology, we expect that $H^*(C^\infty(M)[[\hbar]], C^\infty(M)[[\hbar]]) \cong H^*(M)[[\hbar]]$. Under such an isomorphism, the Gerstenhaber bracket on Hochschild cohomology would go to zero on the right hand side. Moreover, the isomorphism at the second order should provide an intrinsic explanation for the characteristic class of a $*$ -algebra ([27]).

(2). Using the identification as in Theorem 8.2, one obtains a Lie bracket on the space $Z^1(M)[[\hbar]]$ such that the bracket of any closed one-forms is exact (see Proposition 8.5). It is easy to see that for any θ_1, θ_2 , $[\theta_1, \theta_2] = \{\theta_1, \theta_2\} + O(\hbar)$, where $\{\cdot, \cdot\}$ refers to the standard Lie bracket on one-forms induced from the Poisson bracket (see [1]). However, it is difficult to find an explicit expression for the entire bracket $[\cdot, \cdot]$. The latter should be related to the Weyl curvature of the deformation.

Also, for a symplectic manifold, it is well known that the Poisson bracket defined on closed one-forms extends to a bracket on all one-forms. It is not clear, however, whether one can extend the bracket $[\cdot, \cdot]$ above to $\Omega^1(M)[[\hbar]]$. It seems that these problems are all related to the question raised by Weinstein regarding “quantum Lie algebroids” [26].

Another interesting consequence of Theorem 8.2 is the following:

Corollary 8.6 *If $\delta = D_0 + \hbar D_1 + \hbar^2 D_2 + \cdots = \sum_i \hbar^i D_i$ is a derivation. Then both $\delta_{\text{even}} = D_0 + \hbar^2 D_2 + \cdots = \sum_i \hbar^{2i} D_{2i}$ and $\delta_{\text{odd}} = D_1 + \hbar^2 D_3 + \cdots = \sum_i \hbar^{2i} D_{2i+1}$, are derivations.*

Proof. Without loss of generality, we assume that this is a Fedosov $*$ -product, and consider W_D as our algebra.

Then each derivation δ can be written as $\delta = [\frac{i}{\hbar}K, \cdot]$, for some $K \in \Gamma W$. Let $\theta = DK \in Z^1(M)[[\hbar]]$. Write $\theta = \theta_{\text{even}} + \hbar \theta_{\text{odd}}$, where θ_{even} is the sum of all even terms in \hbar while θ_{odd} is the sum of all odd terms in \hbar divided by \hbar . Let K_{even} and K_{odd} be the sections of W corresponding to θ_{even} and θ_{odd} , respectively, given by Equation (40), and let

$$\begin{aligned}\tilde{\delta}_{\text{even}} &= \text{ad}_{\frac{i}{\hbar}K_{\text{even}}} \quad \text{and} \\ \tilde{\delta}_{\text{odd}} &= \text{ad}_{\frac{i}{\hbar}K_{\text{odd}}}.\end{aligned}$$

Then, clearly both $\tilde{\delta}_{\text{even}}$ and $\tilde{\delta}_{\text{odd}}$ are derivations, and consist only even powers of \hbar according to Lemma 8.4. Also $\delta = \tilde{\delta}_{\text{even}} + \hbar \tilde{\delta}_{\text{odd}}$ by construction. Therefore, $\tilde{\delta}_{\text{even}} = \delta_{\text{even}}$ and $\tilde{\delta}_{\text{odd}} = \delta_{\text{odd}}$. This concludes the proof. □

9 Appendix B

A $*$ -product

$$f *_\hbar g = \sum_k \hbar^k C_k(f, g)$$

on a regular Poisson manifold is said to be *regular* if $C_k(\cdot, \cdot)$ is a leafwise bidifferential operator for every k .

In this section, for completeness, we will outline a proof for the following result, whose proof can also be found in [20].

Proposition 9.1 *Suppose that P is a regular Poisson manifold whose symplectic foliation is a fibration $P \rightarrow M$. Assume that the second Betti number of the fibers is zero. Then there exists essentially a unique regular $*$ -product on P . I.e., any two regular $*$ -products are equivalent, and the equivalence operator can be chosen as a formal power series in \hbar with coefficients being leafwise differential operators.*

Recall that, given a fibration \mathcal{F} on a manifold P , a leafwise Hochschild cochain is a k -linear form on $C^\infty(P)$ with value in $C^\infty(P)$, which requires to be a leafwise k -differential operator on P . The Hochschild differential is given by

$$\begin{aligned}(\delta c)(u_0, \dots, u_k) &= u_0 c(u_1, \dots, u_k) + \sum_{i=0}^{k-1} (-1)^{i+1} c(u_0, \dots, u_i u_{i+1}, \dots, u_k) \\ &\quad + (-1)^{k+1} c(u_0, \dots, u_{k-1}) u_k.\end{aligned}$$

As usual, the space of k -cochains is denoted by $C_{\mathcal{F}}^k(C^\infty(P), C^\infty(P))$, while its cohomology is denoted by $H_{\mathcal{F}}^k(C^\infty(P), C^\infty(P))$.

Lemma 9.2 (*Nest-Tsygan [20]*)

Let \mathcal{F} be a fibration on a manifold P . Then,

$$H_{\mathcal{F}}^k(C^\infty(P), C^\infty(P)) \cong \Gamma(\wedge^k T\mathcal{F}). \quad (41)$$

An immediate consequence is the following

Lemma 9.3 (i). If $c \in C_{\mathcal{F}}^2(C^\infty(P), C^\infty(P))$ is an antisymmetric two-cocycle, then c is a leafwise bivector field, i.e., $c \in \Gamma(\wedge^2 T\mathcal{F})$.

(ii). If $c \in C_{\mathcal{F}}^2(C^\infty(P), C^\infty(P))$ is a symmetric two-cocycle, then it is a coboundary.

Proof of Proposition 9.1 The proof is simply a modification of the standard one for symplectic manifolds. The idea, roughly speaking, is as follows. The classification of $*$ -products is equivalent to that of their commutator Lie algebras $[\cdot, \cdot]_*$, which are Lie algebra deformations of the Poisson bracket $\{\cdot, \cdot\}$. The latter is classified by the 2nd order leafwise Chevalley cohomology of the Poisson Lie algebra. However, the 2nd order leafwise Chevalley cohomology is isomorphic to the second leafwise de-Rham cohomology, and is zero by assumption.

Recall that the Chevalley coboundary (see [18]) operator is given by:

$$(\partial c)(u_0, \dots, u_p) = \epsilon_{0 \dots p}^{\lambda_0 \dots \lambda_p} \left[\frac{1}{p!} \{u_{\lambda_0}, c(u_{\lambda_1}, \dots, u_{\lambda_p})\} - \frac{1}{2(p-1)!} c(\{u_{\lambda_0}, u_{\lambda_1}\}, u_{\lambda_2}, \dots, u_{\lambda_p}) \right], \quad (42)$$

where $u_\lambda \in C^\infty(P)$ and ϵ is the Kronecker symbol. If $c \in C_{\mathcal{F}}^2(C^\infty(P), C^\infty(P))$ is antisymmetric,

$$(\partial c)(u_0, u_1, u_2) = (\{u_0, c(u_1, u_2)\} - c(\{u_0, u_1\}, u_2)) + c.p.,$$

where $c.p.$ stands for the cyclic permutation. Hence if $c \in \Gamma(\wedge^2 T\mathcal{F})$, $\partial c = [\pi, c] = d_\pi c$, where the bracket is the Schouten bracket on multivector fields and $d_\pi : \Gamma(\wedge^* T\mathcal{F}) \rightarrow \Gamma(\wedge^{*+1} T\mathcal{F})$ is the differential operator defining the (leafwise)-Poisson cohomology.

Suppose that

$$\begin{aligned} f * g &= \sum \hbar^k C_k(f, g), \quad \text{and} \\ f *' g &= \sum \hbar^k C'_k(f, g) \end{aligned}$$

are two regular $*$ -products on P .

We need to construct an equivalence operator between them. This will proceed by induction. Assume that they are equivalent up to $i = 2k$, i.e., we can find an equivalence operator under which, $C_i = C'_i$ for $0 \leq i \leq 2k$. Then, $\delta(C_{2k+1} - C'_{2k+1}) = 0$. Since $C_{2k+1} - C'_{2k+1}$ is skew-symmetric, it belongs to $\Gamma(\wedge^2 T\mathcal{F})$ by Lemma 9.3.

On the other hand, both $[\cdot, \cdot]_*$ and $[\cdot, \cdot]_{*'}$ are deformations of the Poisson Lie algebra $\{\cdot, \cdot\}$. Thus, $\partial(C_{2k+1} - C'_{2k+1}) = 0$. That is, $[\pi, C_{2k+1} - C'_{2k+1}] = 0$. In other words, $C_{2k+1} - C'_{2k+1}$ is a 2-cocycle of the (leafwise)-Poisson cohomology, which is isomorphic to the leafwise de-Rham cohomology. Since the 2nd leafwise de-Rham cohomology is zero by assumption, there is a vector field $X \in \Gamma(T\mathcal{F})$ such that $C_{2k+1} - C'_{2k+1} = [\pi, X]$. In other words,

$$C_{2k+1}(f, g) - C'_{2k+1}(f, g) = \{Xf, g\} + \{f, Xg\} - X\{f, g\}, \quad \forall f, g \in C^\infty(P).$$

It is then easy to see that $T = 1 + \hbar^{2k}X$ establishes an isomorphism between $*$ and $*'$ up to \hbar^{2k+1} . That is,

$$T(f * g) - Tf *' Tg = O(\hbar^{2k+2}).$$

Finally, we assume that $*$ and $*'$ are equivalent up to $n = 2k - 1$. By applying an equivalence operator, we may assume that $C_i = C'_i$ for $0 \leq i \leq 2k - 1$. Then, $\delta(C_{2k} - C'_{2k}) = 0$. Since $C_{2k} - C'_{2k}$ is a symmetric 2-cochain, $c_{2k} - c'_{2k} = \delta D$, for some $D \in C^1_{\mathcal{F}}(C^\infty(P), C^\infty(P))$ according to Lemma 9.3. Thus $T = 1 + \hbar^{2k}D$ establishes an isomorphism between $*$ and $*'$ up to \hbar^{2k} . This concludes the proof.

□

10 Appendix C

Recall that a *natural* $*$ -product on a symplectic manifold M is a $*$ -product:

$$u *_{\hbar} v = uv - \frac{i\hbar}{2}\{u, v\} + \frac{1}{2}\left(-\frac{i\hbar}{2}\right)^2 Q_2(u, v) + \cdots,$$

where $Q_2(u, v)$ is a bidifferential operator of order 2 in each argument.

It is well known [5] [18] that associated to a natural $*$ -product there is a canonical torsion-free symplectic connection. More precisely, we have

Proposition 10.1 *Let $u *_{\hbar} v = uv - \frac{i\hbar}{2}\{u, v\} + \frac{1}{2}\left(-\frac{i\hbar}{2}\right)^2 Q_2(u, v) + \cdots$ be a natural $*$ -product on M . Then there exists a unique torsion-free symplectic connection ∇ such that*

$$Q_2(u, v) = P_{\nabla}^2(u, v) + H(u, v),$$

where $H(u, v)$ is a bidifferential operator of maximum order 1 in each argument.

For completeness, we outline a proof here in this appendix.

Proof of Proposition 10.1 Take a torsion free symplectic connection $\tilde{\nabla}$. Then,

$$u *_{\hbar} v = uv - \frac{i\hbar}{2}\{u, v\} + \frac{1}{2}\left(-\frac{i\hbar}{2}\right)^2 P_{\tilde{\nabla}}^2(u, v)$$

is a deformation of order up to \hbar^2 . Therefore, $\delta(Q_2 - P_{\tilde{\nabla}}^2) = 0$, where δ denotes the usual Hochschild differential. Since $Q_2 - P_{\tilde{\nabla}}^2$ is symmetric, it must be a 2-coboundary. Hence there is a differential operator D of maximum order 3 such that

$$Q_2 - P_{\tilde{\nabla}}^2 = \delta D.$$

The principal term of D corresponds to a 3-covariant symmetric tensor T^{ijk} .

In local coordinates, write $\tilde{\nabla} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} = \tilde{\Gamma}_{ij}^k \frac{\partial}{\partial x^k}$, and let $\tilde{\Gamma}^{ijk} = \tilde{\Gamma}_{lm}^i \pi^{lj} \pi^{mk}$. Set

$$\Gamma^{ijk} = \tilde{\Gamma}^{ijk} + 3T^{ijk}. \quad (43)$$

Since T^{ijk} is a completely symmetric tensor, the equation above defines a torsion free symplectic connection $\nabla \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} = \Gamma_{ij}^k \frac{\partial}{\partial x^k}$ with $\Gamma^{ijk} = \Gamma_{lm}^i \pi^{lj} \pi^{mk}$. A simple calculation yields that

$$\begin{aligned} P_{\tilde{\nabla}}^2(u, v) - P_{\nabla}^2(u, v) &= \pi^{i_1 j_1} \pi^{i_2 j_2} (\tilde{\Gamma}_{i_1 i_2}^k \tilde{\Gamma}_{j_1 j_2}^l - \Gamma_{i_1 i_2}^k \Gamma_{j_1 j_2}^l) \frac{\partial u}{\partial x^k} \frac{\partial v}{\partial x^l} \\ &\quad + 3T^{i_1 i_2 i_3} \frac{\partial u}{\partial x^{i_1}} \frac{\partial^2 v}{\partial x^{i_2} \partial x^{i_3}} + 3T^{i_1 i_2 i_3} \frac{\partial^2 u}{\partial x^{i_1} \partial x^{i_2}} \frac{\partial v}{\partial x^{i_3}}. \end{aligned} \quad (44)$$

On the other hand,

$$(\delta D)(u, v) = -3T^{i_1 i_2 i_3} \frac{\partial u}{\partial x^{i_1}} \frac{\partial^2 v}{\partial x^{i_2} \partial x^{i_3}} - 3T^{i_1 i_2 i_3} \frac{\partial^2 u}{\partial x^{i_1} \partial x^{i_2}} \frac{\partial v}{\partial x^{i_3}} + \tilde{H}(u, v),$$

where $\tilde{H}(u, v)$ is a bidifferential operator of maximum order 1 in each argument. Therefore,

$$Q_2(u, v) - P_{\tilde{\nabla}}^2(u, v) = P_{\tilde{\nabla}}^2(u, v) - P_{\nabla}^2(u, v) + (\delta D)(u, v)$$

is clearly a bidifferential operator of maximum order 1 in each argument.

To see that such a connection is unique, it suffices to note that $P_{\tilde{\nabla}}^2(u, v) - P_{\nabla}^2(u, v)$ is a bidifferential operator of maximum order 1 iff $\tilde{\nabla} = \nabla$. This can be easily seen from Equation (44).

□

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